Totally disconnected, locally compact groups

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A **topological group** G is a group that is also a topological space, such that $(x, y) \mapsto x^{-1}y$ is continuous. We say G is **locally compact** if it is Hausdorff and there is a compact neighbourhood of 1.

Given any locally compact group G, there is a unique largest connected subgroup G°. The quotient G/G° is then a **totally disconnected locally compact group** (t.d.l.c. group).

Special case: Every totally disconnected *compact* group is a profinite group, that is, an inverse limit of finite groups.

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Special case: Every totally disconnected *compact* group is a profinite group, that is, an inverse limit of finite groups.

Many properties of profinite groups can be read from the finite continuous images. For instance:

Proposition

Let G be a profinite group and let $d \in \mathbb{N}$. Then $G = \overline{\langle X \rangle}$ for X a subset of size $\leq d$ if and only if every finite continuous image of G can be generated by a set of $\leq d$ elements.

Proposition

Let G be a profinite group and let $x, y \in G$. Then x and y are conjugate in G (i.e. there exists $g \in G$ such that $gxg^{-1} = y$) if and only if they are conjugate in every finite continuous image.

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Theorem (Van Dantzig)

Let G be a totally disconnected, locally compact group. Then the compact (= profinite) open subgroups of G form a base of neighbourhoods of the identity.

Lemma

Let G be a topological group and let U and V be open compact subgroups of G. Then U and V are **commensurate**, that is $U \cap V$ has finite index in both U and V.

Corollary

Every non-discrete t.d.l.c. group G has a distinguished **commensurability class** $\mathcal{U}(G)$ of infinite residually finite subgroups, namely its open compact subgroups.

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The previous corollary has a kind of converse: Suppose G is an abstract group with a subgroup U, such that U is **commensurated** in G, that is, gUg^{-1} is commensurate with U for all $g \in G$. Then we can 'complete' (G, U) to obtain a homomorphism $\phi : G \to H$ where

- H is a t.d.l.c. group;
- $\phi(G)$ is dense in H;
- There is a compact open subgroup W of H such that $U = \phi^{-1}(W)$.

Two (in general inequivalent) constructions to obtain a completion of this form were given by Schlichting and Belyaev, and there is a general framework for t.d.l.c. completions in terms of uniformities (R.–Wesolek).

Example: $(\mathrm{SL}_n(\mathbb{Z}[1/p]), \mathrm{SL}_n(\mathbb{Z}))$ can be completed to obtain $\mathrm{SL}_n(\mathbb{Q}_p)$; $\mathrm{SL}_n(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z}[1/p]) \cap \mathrm{SL}_n(\mathbb{Z}_p)$.

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Instead of starting with a group G with a commensurated subgroup U, we could instead start with an abstract or profinite group U and try to construct a group G that contains it as a commensurated subgroup. This works well if U has **icc**, that is, all non-trivial conjugacy classes of U are infinite.

A **virtual automorphism** of a (profinite) group U is an isomorphism between finite index (open) subgroups of U. Two virtual automorphisms are **equivalent** if they admit a common restriction to a finite index (open) subgroup.

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Theorem (Barnea–Ershov–Weigel)

Let U be an icc (profinite) group. Then the set of all equivalence classes of virtual automorphisms of U forms a (t.d.l.c.) group $\mathrm{Comm}(U)$, with a copy of U as a commensurated (compact open) subgroup. Moreover, given any group G containing U as a commensurated subgroup, then there is a canonical homomorphism $\phi : G \to \mathrm{Comm}(U)$, with the kernel consisting exactly of those elements of G that centralize a finite index subgroup of U.

Example: Let T be the binary rooted tree (vertices $\leftrightarrow$ finite binary sequences) and let $U = \mathrm{Aut}(T)$. Then U is a profinite group; $N = \mathrm{Comm}(U)$ (**Neretin's group**) is the group of 'piecewise homotheties' of ∂T, where ∂T is the boundary of T equipped with the natural metric.

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Let X be an infinite set. We put a topology on $\mathrm{Sym}(X)$ (the **permutation topology**) with a base given by sets of the form

$$\{g \in \mathrm{Sym}(X) \mid \forall y \in Y : \ g(y) = f(y)\}$$

where Y is a finite subset of X and $f \in \mathrm{Sym}(X)$.

$\mathrm{Sym}(X)$ is totally disconnected, and many closed subgroups are also locally compact (hence t.d.l.c.):

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Let G be a closed subgroup of $\mathrm{Sym}(X)$.

- G is compact if and only if it has finite orbits on X.
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Theorem (Cameron)

Let G be a subgroup of $\mathrm{Sym}(X)$. Then G is closed if and only if G is the automorphism group of a first-order structure (= collection of relations of any finite arity) on X.

Examples of first-order structures: graphs, simplicial complexes, incidence geometries, designs, algebraic varieties, matroids, groups, rings, modules...

The automorphism group of such a structure will be t.d.l.c. as long as there is some ‘local finiteness’ condition that ensures finite orbits once we fix enough points. It will be non-discrete as long as automorphisms aren’t determined by the image of a given finite set of points.

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In particular, the automorphism group of a connected locally finite graph $\mathfrak{g}$ is a t.d.l.c. group. (Connected + locally finite $\Rightarrow$ every vertex stabilizer has finite orbits.) If a closed subgroup G of $\mathrm{Aut}(\mathfrak{g})$ has finitely many orbits on the vertices, then it can be shown that G is generated by a **compact** subset.

There is a (quasi-)converse to this fact:

Theorem (Abels)

Let G be a compactly generated t.d.l.c. group and let U be an open compact subgroup of G. Then G acts on a locally finite graph, with vertex set corresponding to the set of left cosets of U in G.

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Call a graph as described by Abels' theorem a **Cayley–Abels graph** for G. Given a locally compact group $G = \langle X \rangle$ where X is compact, then the word metric on G with respect to X only depends on the choice of X up to quasi-isometry: in other words, given two such metrics d_1 and d_2, there is a constant C such that

$$\frac{d_1(x,y)}{C} - C \leq d_2(x,y) \leq Cd_1(x,y) + C.$$

More generally, there is a good analogue of the theory of Cayley graphs for finitely generated groups, and related notions like hyperbolicity, number of ends and so on. (See 'Metric geometry of locally compact groups' by Cornulier and de la Harpe.)

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A linear algebraic group $G = \mathbb{G}(K)$ over a field K is a group of $n \times n$ matrices over K specified by polynomial equations, for example, the special linear group ($\det(A) = 1$) or the orthogonal group ($A^{-1} = A^t$). If K is a topological field, then the set of $n \times n$ matrices over K carries a topology (product topology of $K^{n \times n}$), and then $\mathbb{G}(K)$ is a topological group with the subspace topology.

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If instead K is $\mathbb{Q}_p$, $\mathbb{F}_q((t))$ or some finite extension of these, then K is a non-Archimedean local field: it has a largest compact subring O_K (the **valuation ring**), and there is a unique largest proper ideal m of O_K, such that the residue field $k = O_K/m$ is finite. (For example: $K = \mathbb{F}_q((t))$, $O_K = \mathbb{F}_q[[t]]$, $m = t\mathbb{F}_q[[t]]$, $k = \mathbb{F}_q$.)

A linear algebraic group $G = \mathbb{G}(K)$ is then a t.d.l.c. group: it has a compact open subgroup $\mathbb{G}(O_K)$ and a base of neighbourhoods of the identity of given by the compact open subgroups U_n, where U_n consists of all elements A such that $A - I$ has coefficients in m^n ($m^0 = O_K$).

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Simple algebraic groups over non-Archimedean local fields have a rich structural theory and were the first family of non-compact, non-discrete t.d.l.c. groups to be studied in detail. Every such group has two associated combinatorial structures (an affine building, with a spherical building as its boundary) on which the group acts; moreover all affine and spherical buildings of high enough dimension arise in this way (Tits et al).

A similar structure theory also applies to the simple groups of Lie type, that is, simple (twisted) algebraic groups over finite fields. In both cases, there is a complete classification of the simple groups that arise.

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For both finite groups and connected Lie groups, there is a well-developed structure theory:

- There is a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

  of closed subgroups, where each G_i/G_{i+1} is simple or abelian;
- The simple groups in the class have been classified;
- There is an extensive literature on soluble (finite/Lie) groups.

Profinite groups and connected locally compact groups are inverse limits of finite and connected Lie groups respectively, and many properties can be understood in these terms.
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Could one develop an analogous structure theory for t.d.l.c. groups?

The t.d.l.c. case presents several additional difficulties:

(i) The class of t.d.l.c. groups includes all groups with the discrete topology, so we need a structure theory that involves the group topology in an essential way.

(ii) There does not appear to be any useful finite invariant like 'dimension' that is monotone on closed subgroups. So we cannot expect such strong finiteness properties as in Lie/algebraic groups.

(iii) Even the class $\mathcal{S}$ of compactly generated nondiscrete topologically simple t.d.l.c. groups has $2^{\aleph_0}$ nonisomorphic groups in it (Smith), with no known strategy to try to classify them up to isomorphism.

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Nevertheless, a detailed theory of t.d.l.c. groups has begun to emerge, with various different approaches:

- Methods based on measure/integration (applicable generally to locally compact groups, topology is not so prominent);
- Geometric group theory;
- Scale theory (analogous to the spectral theory of operators);
- Chabauty space structure and dynamics;
- Decomposition theory;
- Local structure (i.e. based on identity neighbourhoods).

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A natural approach to understanding the structure of the t.d.l.c. group G is in terms of how it is built out of 'simpler' parts. Basic ingredients:

- All compact and discrete groups are t.d.l.c. groups; let's take these as the base case.
- Every t.d.l.c. group is a directed union of compactly generated t.d.l.c. groups, so we can try to understand a t.d.l.c. group via its compactly generated open subgroups.
- One can put a number of restrictions on closed normal subgroups of compactly generated t.d.l.c. groups. However, the closed normal subgroups need not themselves be compactly generated.

For technical reasons it is convenient to restrict to the class of t.d.l.c. second-countable (t.d.l.c.s.c.) groups.

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As remarked earlier, there is no good finite invariant that behaves like 'dimension'. However, there is a large class of t.d.l.c.s.c. groups admitting a robust *ordinal-valued* rank.

Definition (Wesolek)

The class $\mathcal{E}$ of **elementary** t.d.l.c.s.c. groups is the smallest class of t.d.l.c.s.c. groups that contains the second-countable profinite and countable discrete groups, and is closed under the following operations: closed subgroups, Hausdorff quotients, increasing unions of open subgroups, and extensions that result in a t.d.l.c.s.c. group.

Write $\mathrm{Res}(G)$ for the intersection of all open normal subgroups of G (in other words, the intersection of all normal subgroups N such that G/N is discrete); note that if G is profinite or discrete, then $\mathrm{Res}(G) = \{1\}$.

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As remarked earlier, there is no good finite invariant that behaves like ‘dimension’. However, there is a large class of t.d.l.c.s.c. groups admitting a robust *ordinal-valued* rank.

Definition (Wesolek)

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Theorem (Wesolek)

Let G be a t.d.l.c.s.c. group. Then G is elementary if and only if there does not exist an infinite descending chain of compactly generated closed subgroups G_i such that $\forall i : G_{i+1} \leq \mathrm{Res}(G_i)$. (We allow $G_{i+1} = \mathrm{Res}(G_i) = G_i$ here.)

If G is elementary and G is an increasing union of compactly generated open subgroups O_i, there is a canonically defined rank function ξ, taking values in the countable ordinals, characterized by the properties $\xi(G) = 1 \Leftrightarrow G = \{1\}$ and $\xi(G) = \sup\{\xi(\mathrm{Res}(O_i))\} + 1$.

One can show (R.) that if G is elementary, then for every ordinal α such that $\alpha + 2 \leq \xi(G)$, there is a compactly generated open subgroup of rank *exactly* $\alpha + 2$.

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Not all t.d.l.c.s.c. groups are elementary, for instance if G is such that some group in $\mathcal{S}$ appears as a quotient of a subgroup, then G is nonelementary. (The converse is an open problem.)

However, for all compactly generated t.d.l.c. groups, there are strong limitations on the class of closed *normal* subgroups. Say K/L is a **chief factor** of G if K and L are closed normal subgroups of G, such that $K > L$ and no closed normal subgroup of G lies strictly between K and L.

Theorem (R.–Wesolek)

Let G be a compactly generated t.d.l.c. group. Then G has an **essentially chief series**, meaning a finite series

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Theorem (R.–Wesolek)

Let G be a topologically characteristically simple t.d.l.c. group. Then at least one of the following holds:

(i) (Semisimple type) G is a quasi-product of simple groups;

(ii) ("Stacking type") There is a characteristic class $\mathcal{K}$ of chief factors K/L of G, such that for any $K_1/L_1, K_2/L_2 \in \mathcal{K}$, there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(K_1) < L_2$;

(iii) $\xi(G) \in \{2, \omega + 1\}$.

So if G is nonelementary, it has a nonelementary chief factor of one of the two types above. In case (i) we reduce to nonelementary simple groups (not necessarily compactly generated). In case (ii), the chief factor itself has nonelementary chief factors and we can repeat the argument. (Open problem: if we keep repeating the argument, do we arrive at a semisimple chief factor?)

Say t.d.l.c. groups G and H are **locally isomorphic** if there is an isomorphism from an open subgroup of G to one of H; this defines an equivalence relation on t.d.l.c. groups. A **local property** of G is one that is determined by its local isomorphism type.

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The **quasi-centre** $\mathrm{QZ}(G)$ of a t.d.l.c. group G is the set of elements with open centralizer. If U is profinite, then $\mathrm{QZ}(U) = \{1\} \Leftrightarrow U$ has icc.

Theorem (Caprace–R.–Willis)

Let U be a profinite group with $\mathrm{QZ}(U) = \{1\}$. Let $\mathcal{LD}(U)$ be the set of commensurability classes of direct factors of open subgroups of U. Then $\mathcal{LD}(U)$ is a Boolean algebra and a local property of U.

Since $\mathcal{LD}(U)$ is a local invariant, we can extend the definition to t.d.l.c. groups, where $\mathcal{LD}(G) = \mathcal{LD}(U)$ for any compact open subgroup U of G. Then $G \curvearrowright \mathcal{LD}(G)$ by conjugation.

Say G is **faithful locally decomposable** if $\mathrm{QZ}(G) = \{1\}$ and G acts faithfully on $\mathcal{LD}(G)$. This is a local property amongst groups with trivial quasi-centre.

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- ‘Most’ known groups in $\mathcal{S}$ are faithful locally decomposable. (Exceptions: algebraic groups, Kac–Moody groups, and certain groups acting on right-angled buildings.)
- (Garrido–R.–Robertson) Every t.d.l.c.s.c. group with faithful piecewise full action on the Cantor set (e.g. Neretin’s groups) is faithful locally decomposable; conversely, any t.d.l.c.s.c. group that is faithful locally decomposable is an open subgroup of a piecewise full group on the Cantor set.
- (Caprace–R.–Willis) If $G \in \mathcal{S}$ is faithful locally decomposable, then the action of G on the Stone space of $\mathcal{LD}(G)$ is minimal and extremely proximal. Moreover, G is nonamenable, abstractly simple, and has an element with nontrivial contraction group. (Open problem: do all groups in $\mathcal{S}$ have the last three properties?)

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