Recent developments on edge-transitive graphs and maps

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Graphs and their automorphisms

In this talk, every graph will be finite and undirected, and usually simple (with no loops or multiple edges).

An automorphism (or symmetry) of a graph X is defined as a permutation of both the vertices and edges that preserves adjacency and incidence. Under composition, these permutations form a group called the automorphism group of X, and denoted by $\mathrm{Aut}(X)$.

Examples

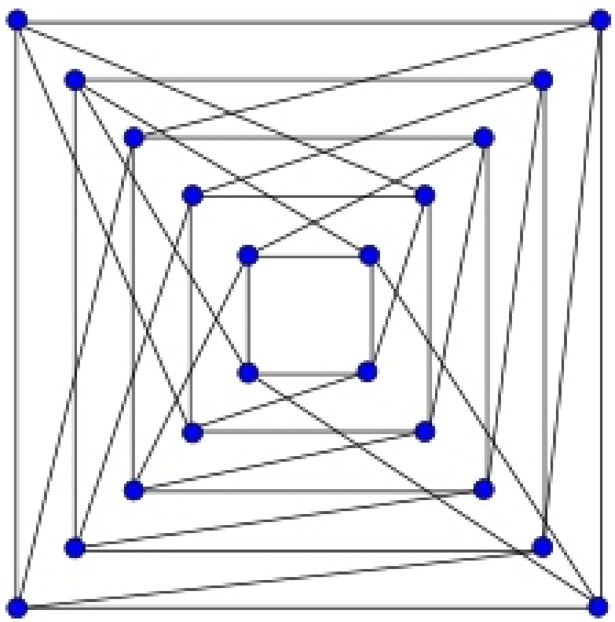
- Complete graphs: $\mathrm{Aut}(K_n) \cong S_n$, of order $n!$
- Cycle graphs: $\mathrm{Aut}(C_n) \cong D_n$, dihedral of order $2n$
- Path graphs: $\mathrm{Aut}(P_n) \cong C_2$ when $n \geq 2$

Edge-transitive graphs

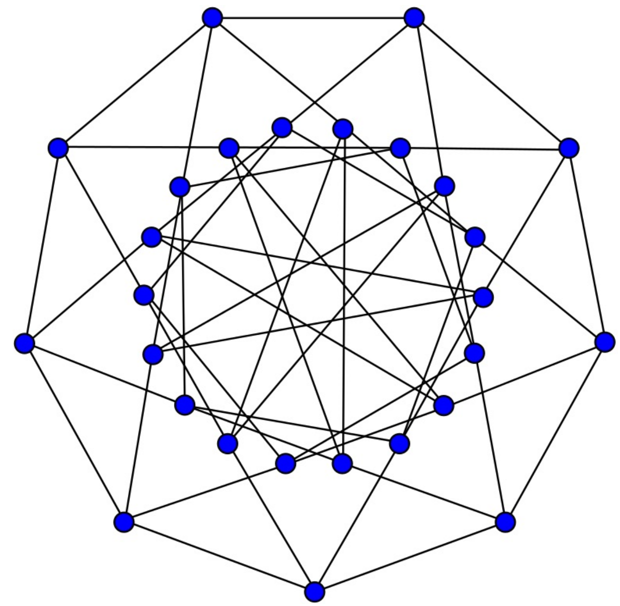
A graph is edge-transitive (ET) if its automorphism group has a single orbit on edges. Such a graph might or might not be also vertex-transitive (VT), or arc-transitive (AT).

Examples

- Cycle graphs C_n are ET, VT and AT
- Complete graphs K_n are ET, VT and AT
- Complete bipartite graphs $K_{m,n}$ are all ET, but are VT (and AT) only when $m = n$
- Semi-symmetric graphs are regular and ET but not VT (e.g. the Folkman and Gray graphs).
- Half-arc-transitive graphs are ET and VT but are not AT (e.g. the Doyle-Holt graph of order 27).



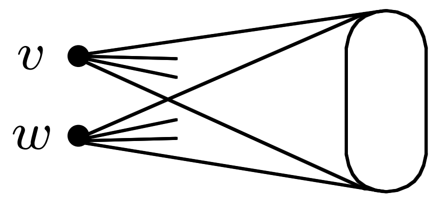
Folkman graph



Doyle-Holt graph

Important digression: Twin-free graphs

Two vertices in a graph are called twins [thanks to Kotlov and Lovász (1996)] if they have the same neighbours:



In this case, there exists an automorphism of the graph that swaps v and w but fixes all others. This can make the automorphism group quite large (but imprimitive).

Examples include the complete bipartite graphs $K_{m,n}$ for $\max(m, n) \geq 2$, with automorphism group $S_m \times S_n$ or $S_n \wr C_2$.

If there is no such pair $\{v, w\}$, then the graph is twin-free (or 'worthy', according to Steve Wilson).

Finding vertex-transitive graphs

It is fairly straightforward to find all small vertex-transitive graphs using the list of all transitive groups of small degree. These groups and graphs are known for degree n up to 47. For some time, those of degree 32 were a road-block, til recent work by Derek Holt (Warwick) & Gordon Royle (UWA).

For a vertex-transitive graph X of order n, let $A = \mathrm{Aut}(X)$. Then A acts transitively and faithfully on $V(X)$, and for any vertex v, the neighbourhood $X(v)$ is a union of orbits of the vertex-stabiliser A_v on $V(X) \setminus \{v\}$.

Conversely, if P is any transitive group of degree n, and Δ is a union of orbits of the point-stabiliser P_1, then the union of the P-orbits of the pairs $\{1, \delta\}$ with $\delta \in \Delta \setminus \{1\}$ gives the edge-set of a graph X of order n on which P acts vertex-transitively. (Note: P need not be all of $\mathrm{Aut}(X)$.)

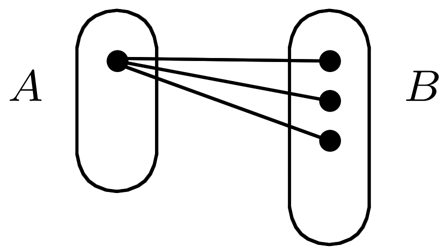
What about edge-transitive graphs?

To find all small edge-transitive (ET) graphs, it's helpful to consider separately the two cases according to whether or not the graph is also vertex-transitive (VT).

It turns out that the latter case helps with the former case when the graph is bipartite.

Case (a): **Edge- but not vertex-transitive graphs**

In this case Aut(X) has two orbits on $V(X)$, say A and B, with every edge of X joining a vertex of A to a vertex of B. Hence in particular, X is bipartite.



Also X is locally arc-transitive (i.e. the stabiliser in Aut(X) of any vertex v is transitive on the arcs of the form (v, w)).

Next, we may suppose X is twin-free since every edge-transitive graph with twins can be constructed as a 'blow-up' of a twin-free example. [Why/how? See next slide]

‘Blow-ups’ of ET graphs

Let X be a bipartite graph, with parts A and B (say).

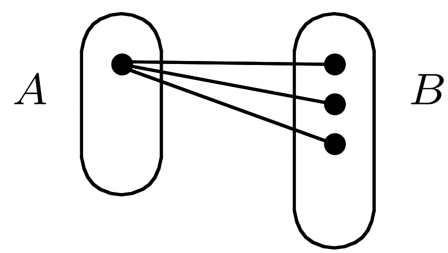
Then for any pair (m, n) of positive integers, replace every vertex $a \in A$ by m new vertices $a_1, a_2, \ldots, a_m$ and every vertex $b \in B$ by n new vertices $b_1, b_2, \ldots, b_n$, and replace every edge $\{a, b\}$ by mn edges $\{a_i, b_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

This gives the (m, n) blow-up of X.

For example, the (m, n) blow-up of K_2 is the graph $K_{m,n}$.

If X is edge-transitive, then so are all of its blow-ups. And conversely, if Y is an ET bipartite graph with twins, then Y is a (unique) blow-up of a twin-free ET bipartite graph X, each of whose vertices corresponds to a maximal set of vertices of Y having the same neighbourhood (in Y).

Moreover, when X is a <u>twin-free</u> ET bipartite graph (with parts A and B), then $G = \mathrm{Aut}(X)$ acts faithfully and transitively on each of the parts A and B of X, and hence we can think of G as a transitive permutation group on A with an auxiliary faithful and transitive action on B. In particular, the neighbours in B of a vertex $a \in A$ form an orbit of G_a in its action on B:



This gives a way to find all such graphs: For every transitive group G on a set A of given degree $|A|$, find all faithful transitive actions of G on a set B of given degree $|B|$, and for each one, construct the bipartite graph whose edges are the pairs in an orbit of G on $A \times B$.

Algorithm

To find all twin-free edge-transitive bipartite graphs with parts of sizes m and n:

(1) Find all transitive permutation groups of degree m

(2) For each such group P, find all faithful transitive permutation representations of P of degree n

(3) For each such representation, and each orbit of the point-stabiliser P_1, check the resulting bipartite graph for connectedness, worthiness, A- and B-valencies, isomorphism with earlier graphs found, vertex-transitivity, and so on.

Note: Step (2) can be taken by finding all conjugacy classes of core-free subgroups of index n in P, and simplified by eliminating subgroups that will not give a connected graph.

Implementation ... not so easy

No problem for 'small' part-sizes m and n, but ...

The numbers of transitive permutation groups of degrees 24, 32, 36 and 40 are 25000, 2801324, 121279 and 315842, respectively. In particular, this makes the cases in which $m, n \in \{24, 32, 36, 40\}$ quiet challenging, and the case where $(m, n) = (32, 32)$ close to ridiculous.

For other cases with $m = 32$ we simply swapped m and n. Even so, the cases $(m, n) = (24, 24)$, $(24, 32)$ and $(24, 36)$ required some special tricks, such as considering interactions of subgroups of P with a normal Hall subgroup of P.

Result: All twin-free bipartite locally arc-transitive graphs of order up to 63, with blow-ups giving all bipartite locally arc-transitive graphs with twins as well.

Case (b): **Edge- and vertex-transitive graphs**

In this case, we can take the standard approach (using the database of transitive permutation groups of small degree) to construct all vertex-transitive graphs of order up to 47, and then check which ones are edge-transitive, etc.

[Note: The transitive groups of degree 48 have *only recently* ~~not yet~~ been determined, so 47 is a hard upper limit at this stage.]

But we can go further, in the bipartite case ...

• All arc-transitive bipartite graphs of order up to 63 can be found in the same way (and at the same time) as those that are edge- but not vertex-transitive, and

• All half-arc-transitive bipartite graphs of order up to 63 can be found in a similar way, by taking two equal-sized orbits of the stabiliser P_1 on the auxiliary set B.

Computation time: Several days (using MAGMA)

Result: All edge-transitive graphs of order up to 47, and all bipartite edge-transitive graphs of order 48 to 63 as well.

Summary:

- 1894 ET graphs of order up to 47
  - 1429 bipartite, 465 non-bipartite
  - 625 twin-free, 1269 with twins
  - 678 vertex-transitive, 1216 not vertex-transitive
- 3312 bipartite ET graphs of order up to 63
  - 795 twin-free, 2517 with twins
  - 438 vertex-transitive, 2874 not vertex-transitive.

Questions by Folkman

At the end of his pioneering 1967 paper on semi-symmetric graphs, Folkman asked eight questions.

Two of these were general questions about the orders and valencies of semi-symmetric graphs, and they remain open.

Another three were about the existence of semi-symmetric graphs of order $2n$ and valency d where d is prime, or d is coprime to n, or d is a prime that does not divide n, and these have been answered by the construction of semi-symmetric 3-valent graphs (including examples of orders 110 and 112).

Another question asked if there is a semi-symmetric graph of order 30, and this was answered by Ivanov, who proved in 1987 that no such graph exists.

Another question asked if there is a semi-symmetric graph of order $2pq$ where p and q are odd primes such that $p < q$, and p does not divide $q-1$. This was answered by Du and Xu, who (in 2000) found all semi-symmetric graphs of order $2pq$ where p and q are distinct primes, and these include graphs of orders $70 = 2 \cdot 5 \cdot 7$, $154 = 2 \cdot 7 \cdot 11$ and $3782 = 2 \cdot 31 \cdot 61$.

The remaining question: Does there exist a semi-symmetric graph of order $2n$ and valency d where $d \geq n/2$?

Our computations answer this positively by producing a few examples, including some with $(2n, d) = (20, 6)$, $(24, 6)$ and $(36, 12)$, giving ratio $d/n = 3/5$, $1/2$ and $2/3$.

Some of these point the way to a construction that proves something even stronger ...

Construction 1

For every integer $r \geq 3$, let A be the union of two disjoint sets A_1 and A_2 of size r, and let $B = A_1 \times A_2$, and make these the parts of a bipartite graph X in which an edge joins each $(a_1, a_2) \in B$ to every $a \in A_1 \setminus \{a_1\}$ and every $a \in A_2 \setminus \{a_2\}$.

Then X has $2r+r^2$ vertices and $2r^2(r-1)$ edges: each $a \in A$ has valency $r(r-1)$ and each $b \in B$ has valency $2(r-1)$.

Also the automorphism group of the graph is isomorphic to the wreath product $S_r \wr C_2$, of order $2(r!)^2$, and this acts transitively on the edges of X, so X is ET.

Now take the $(r, 2)$ blow-up of X. The result is an edge-transitive regular graph of order $4r^2$ and valency $2r(r-1)$, but it is not vertex-transitive, since $r > 2$. Hence this graph is semi-symmetric, with ratio $d/n = 2r(r-1)/(2r^2) = (r-1)/r$.

This gives **an even better answer to Folkman's question**:

> Construction 1 gives an infinite family of semi-symmetric graphs of order $2n = 4r^2$ and valency $d = 2r(r-1)$ in which the ratio $d/n = (r-1)/r$ can be arbitrarily close to 1.

[Note: In case you're wondering, the ratio for the complete bipartite graph $K_{n,n}$ is $n/n = 1$, but this graph is not semi-symmetric: it is arc-transitive.]

Complaint: The semi-symmetric graphs in Construction 1 (and all other examples of order less than 63) have twins!

Question: Are there any <u>twin-free</u> examples with large d/n?

Construction 2 (using some geometry)

For every odd prime power $q \geq 3$, let Q be the generalised quadrangle associated with a symplectic form on $V = \mathbb{F}_q^4$ (such as $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$).

Then Q has q^3+q^2+q+1 points (1-dimensional subspaces of V) and q^3+q^2+q+1 isotropic lines (2-dimsl subspaces U of V with $U = U^\perp$), and the associated Levi/incidence graph is locally arc-transitive, but not vertex-transitive (since the geometry is not self-dual, by a theorem of Benson (1970)).

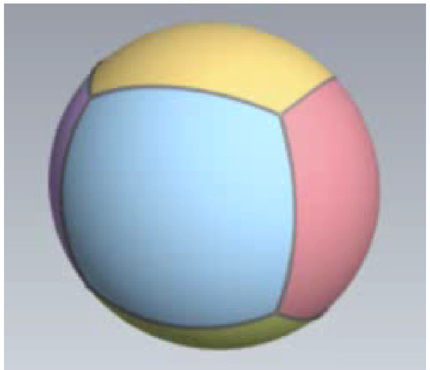
Now take the bipartite complement, in which each point is joined to each of the isotropic lines that do <u>not</u> contain it. This graph is also semi-symmetric, with valency $q^3 + q^2$, and its automorphism group is Aut($PSp(4,q)$), which acts primitively on both parts. Hence the graph is twin-free.

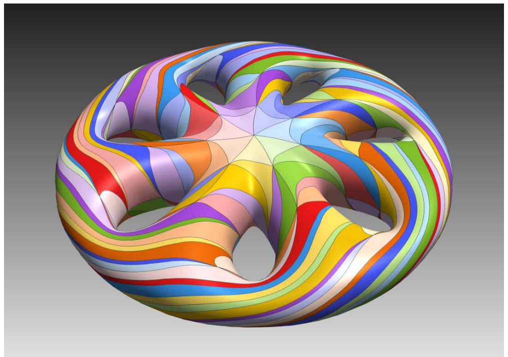
This gives a more satisfying answer to Folkman's question:

Construction 2 gives an infinite family of twin-free semi-symmetric graphs of order $2(q^3+q^2+q+1)$ and valency q^3+q^2 in which the ratio $d/n = (q^3+q^2)/(q^3+q^2+q+1)$ can be arbitrarily close to 1.

Edge-transitive maps

A map M is an embedding of a connected graph X into a closed surface S, breaking up S into simply connected regions called the faces of M.





An automorphism of a map M is a bijection from M to M preserving incidence (between vertices, edges and faces), and then M is called edge-transitive if its automorphism group has a single orbit on edges.

Classification of edge-transitive maps

In a long paper in 1997, Graver and Watkins showed there are 14 different classes of edge-transitive maps. Each class can be defined by what happens 'locally' around a given edge e (and edges incident with e), or by a universal group.

The principal class '1' consists of regular maps, with group

$$\mathcal{U} = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = 1 \,\rangle$$

where $\langle a, b\rangle$, $\langle a, c\rangle$ and $\langle b, c\rangle$ represent the pre-images in $\mathcal{U}$ of stabilisers of an incident vertex, edge and face, respectively. The universal group for the class '2^Pex' of chiral maps is isomorphic to the subgroup generated by $x = ab$ and $y = bc$, with presentation $\langle\, x, y \mid (xy)^2 = 1 \,\rangle$.

The universal groups of all 14 classes are subgroups H of index 1, 2 or 4 in $\mathcal{U}$ with the property that $\mathcal{U} = H\langle a, c\rangle$.

Class	$\lvert \mathrm{Aut}(M) \rvert$	VT	FT	PT	Comments
1	$4\lvert E(M)\rvert$	Yes	Yes	Yes	Fully regular maps
2	$2\lvert E(M)\rvert$	No	Yes	Yes	
2^*	$2\lvert E(M)\rvert$	Yes	No	Yes	Dual to class 2
2^P	$2\lvert E(M)\rvert$	Yes	Yes	No	Petrie dual to class 2
2ex	$2\lvert E(M)\rvert$	Yes	Yes	Yes	
2^*ex	$2\lvert E(M)\rvert$	Yes	Yes	Yes	Dual to class 2ex
2^Pex	$2\lvert E(M)\rvert$	Yes	Yes	Yes	Chiral maps
3	$\lvert E(M)\rvert$	No	No	No	
4	$\lvert E(M)\rvert$	No	Yes	Yes	
4^*	$\lvert E(M)\rvert$	Yes	No	Yes	Dual to class 4
4^P	$\lvert E(M)\rvert$	Yes	Yes	No	Petrie dual to class 4
5	$\lvert E(M)\rvert$	No	Yes	Yes	
5^*	$\lvert E(M)\rvert$	Yes	No	Yes	Dual to class 5
5^P	$\lvert E(M)\rvert$	Yes	Yes	No	Petrie dual to class 5

VT = vertex-transitive, FT = face-transitive, PT = Petrie-transitive

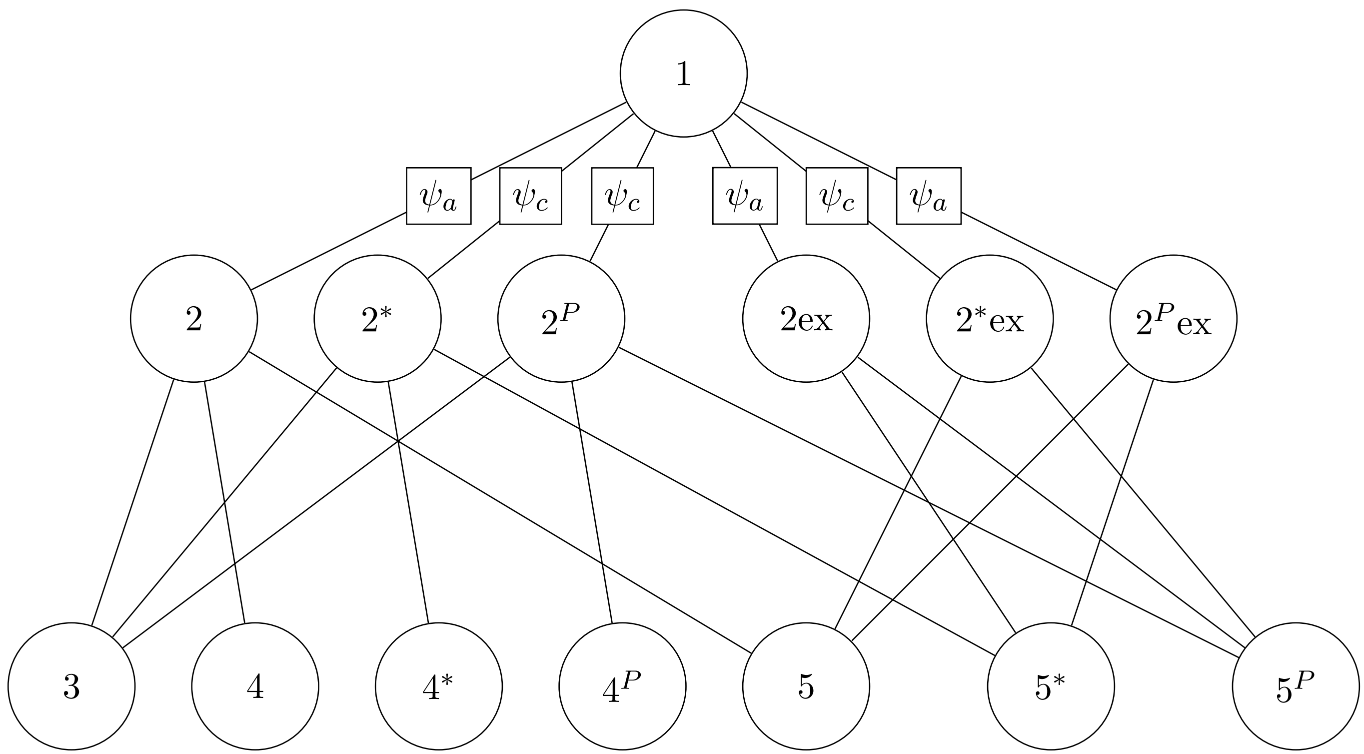
Notes on constructing edge-transitive maps

The automorphism group Aut(M) of an ET map M of any given class c is a non-degenerate homomorphic image of the corresponding universal group $\mathcal{U}_c$ – say $\mathcal{U}_c/K$ for some K.

In that case the map M can be constructed using cosets of the images of relevant subgroups of $\mathcal{U}_c$.

Conversely, if $G = \mathcal{U}_c/K$ is a non-degenerate finite homomorphic image of $\mathcal{U}_c$, then G is an arc-transitive group of automorphisms of an ET map M, but the full automorphism group of M could be larger than G. Ensuring that $G = \text{Aut}(M)$ requires ruling out the possibility that the map M admits certain 'barred automorphisms', or equivalently, that K is normal in one of the 'larger' universal groups.

Inclusions among the 14 universal groups



Questions about edge-transitive maps

The initial work on this topic by Graver and Watkins was taken further by Širáň, Tucker and Watkins (2001), giving examples of ET maps all 14 types — including examples of each type with automorphism groups isomorphic to S_n for all $n \geq 11$ such that $n \equiv 3$ or 11 mod 12, etc.

They also posed several questions at the end of their paper — e.g. about finding the smallest genus of surfaces carrying an ET map of given type, finding all examples for genus 2, and determining the genera of surfaces carrying at least one 'non-degenerate' ET map M (with no loops or multiple edges in the underlying graph of M or its dual).

Some of these questions were answered by Alen Orbanić in his PhD thesis (Ljubljana, 2006). Others were left open.

Answers to some questions about ET maps

1) Which of the 14 classes contain self-dual maps?

This question was posed by Širáň, Tucker & Watkins (2001): Does each of the six classes 1, 2^P, 2^Pex, 3, 4^P and 5^P contain some self-dual map? (None of other eight classes allows self-duality.)

Yes! In each of the classes 1, 2^P, 2^Pex, 3, 4^P and 5^P, there exist self-dual edge-transitive orientable maps such that the map and its dual have simple underlying graph. [MC (2019)]

2) Does some surface carry ET maps of all 14 types?

This question was posed by Širáň, Tucker & Watkins (2001). Note that if it happens, the surface must be orientable.

Yes! There is one of each type on a surface of genus 14. [MC (in 2017), although Alen Orbanić got very close to it]

In fact, on each of the orientable surfaces of genus 17 and 21, there exists an ET map of each of the 14 types, with the additional property that both the map and its dual have simple underlying graph [MC (2019)].

3) Does some graph underly an ET map of each type?

This was another question posed by Širáň, Tucker & Watkins.

In an early search we found that the complete bipartite graph $K_{8,8}$ underlies ET maps of 11 of the 14 classes, namely all of them except classes 2*ex, 2^Pex and 5. A little further work led us to a positive answer to the question:

Yes! The complete bipartite graph $K_{16,16}$ underlies an ET map of each of the 14 types. [MC (2019)]

4) Is some finite group the automorphism group of some ET map in each of the 14 classes?

This was a question raised by Tom Tucker in 2017 (en route to a conference in Slovakia).

Yes! Gareth Jones (2018) showed that a given non-abelian simple group is the automorphism group of some ET map of a given class unless it appears in a table of known exceptions.

In particular, the smallest simple group that occurs as the automorphism group of some ET map in every one of the 14 classes is the Suzuki group $Sz(8)$, of order 29120. (In fact, the map can be chosen such that both it and its dual have simple underlying graph.) Furthermore ... [*next slide*]

• The alternating group A_n is the automorphism group of some ET map in each class, except a few cases when $n \leq 8$. The smallest one that occurs for all 14 classes is A_9. Non-degenerate ET maps occur in all classes for A_9 (and A_{10}).

• The symmetric group S_n is the automorphism group of some ET map in each class, except a few cases when $n \leq 5$. The smallest one that occurs for all 14 classes is S_6 – which is also the smallest insoluble group that occurs in this way. Non-degenerate ET maps occur in all classes for S_6 (to S_{10}).

• The smallest finite group that occurs as the automorphism group of some ET map in each of the 14 classes is a unique (soluble) group of order 576, and non-degenerate ET maps occur in all classes for this group. [MC (2019)]

5) Does every closed orientable surface support some non-degenerate, edge-transitive map?

This was perhaps the most interesting question posed by Širáň, Tucker & Watkins in their paper in 2001.

Yes! For every $g \geq 2$, there exists an edge-transitive map of class 2 on the orientable surface of genus g, such that both the map and its dual have simple underlying graph.

[Answered by MC & honours student Isabel Holm (2018)]

Note that the answer for the analogous question for (fully) regular maps (class 1) is 'No', while the answer for orientably-regular maps (class 1 <u>or</u> 2^Pex) is still not known.

Construction [MC & Isabel Holm]

The universal group $\mathcal{U}_1$ for ET maps of class 1 is the group

$$\mathcal{U}_1 = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = 1 \,\rangle$$

with $\langle a, b\rangle$, $\langle a, c\rangle$ and $\langle b, c\rangle$ giving the stabilisers of a mutually incident face, edge and vertex, respectively.

The universal group $\mathcal{U}_2$ for ET maps of class 2 is the subgroup generated by $(x, y, z) = (b, c, aba)$, with presentation

$$\mathcal{U}_2 = \langle\, x, y, z \mid x^2 = y^2 = z^2 = 1 \,\rangle.$$

Computations using MAGMA give quotients of $\mathcal{U}_2$ that produce examples of ET maps of class 2 having small genus. For many among these, the images of (x, y, z) satisfy the extra relations $[x, z] = (xyzy)^2 = 1$. The relations enabled us to construct an infinite family with the desired properties.

Take $\mathcal{G} = \langle\, x, y, z \mid x^2 = y^2 = z^2 = [x, z] = (xyzy)^2 = 1 \,\rangle$.

In this group, the subgroup N generated by $r = (xy)^2 = [x, y]$ and $s = (yz)^2 = [y, z]$ is normal, with $r^x = r^{-1}$, $r^y = r^{-1}$ and $r^z = r$, and $s^x = s$, $s^y = s^{-1}$ and $s^z = s^{-1}$, with quotient $\mathcal{G}/N \cong (C_2)^3$. Moreover, by Reidemeister-Schreier, this normal subgroup N is free abelian of rank 2.

Now for any even positive integers k and m with $k \neq m$, we can factor out the subgroup generated by $r^{k/2} = (xy)^k$ and $s^{m/2} = (yz)^m$ and obtain a quotient G of order $2km$ that is the automorphism group of a bipartite ET map of class 2.

The underlying graph is isomorphic to the complete bipartite graph $K_{m,k}$ and faces have length 4 (so the dual map is simple too), and the map has genus $g = (k-2)(m-2)/4$, which can be any integer greater than 1 (when $m = 4$).

Finally ... an advertisement ...

A 3rd conference on Symmetries of Discrete Objects will be held the week 10-14 February 2020 in Rotorua, New Zealand

See www.math.auckland.ac.nz/~conder/SODO-2020

All welcome!

THANK YOU FOR LISTENING!