Topos theoretic aspects of self-similarity

Symmetry in Newcastle

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5 April 2019

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A crash course in topos theory

Warm-up: locales

For each topological space X we can consider its lattice of open sets $\mathcal{O}(X)$: a complete lattice satisfying the infinite distributive law:

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i) \; . \qquad (*)$$

A complete lattice satisfying (*) is called a *locale*.

It's possible to re-do all of general topology taking locales as the basic notion in place of topological spaces. Some care is required to phrase everything in terms of open sets, rather than points.

This is called *constructive* or *point-free* topology.

Grothendieck toposes

Locales are slightly generalised topological spaces; Grothendieck toposes are *substantially* generalised topological spaces.

- A locale is a poset with finite meets, arbitrary joins, and compatibilities between the two.
- A Grothendieck topos is a category with finite limits, arbitrary colimits, and compatibilities between the two.

Note: just like the elements of a locale, the objects of a topos represent the *opens* of a generalised space, and not the points.

The *points* of a topos are a derived notion: what makes life interesting is that these points may have automorphisms!

Grothendieck toposes

A Grothendieck topos is not just a generalised space. It is also a universe, like the universe of sets, within which one can do mathematics. Thus, in any topos $\mathcal{C}$ one can speak of:

- *Spaces* (= locales) internal to $\mathcal{C}$;
- *k-modules* internal to $\mathcal{C}$ (where k is a real-world ring);
- *Complex numbers* internal to $\mathcal{C}$;
- *Hilbert spaces* internal to $\mathcal{C}$;

and so on. Some care is needed: the axiom of choice may not hold in $\mathcal{C}$ and so, for example, Heine–Borel and Gelfand–Naimark may fail.

Toposes vs groups

Any (discrete) group G gives rise to a topos:

Definition

The *topos of G-sets* $\mathrm{B}G$ is the category with right G-sets as objects, and as maps, equivariant functions.

Conversely, each object A of a topos $\mathcal{C}$ gives rise to a group:

Definition

The *group of automorphisms* of $A \in \mathcal{C}$ is the group

$$\mathrm{Aut}(A) = \{\, f\colon A \to A \mid f \text{ invertible} \,\}\,.$$

Any space (= locale) S gives rise to a topos:

Definition

The *topos of sheaves* $\mathrm{Sh}(S)$ is the category with as objects the local homeomorphisms $A \to S$, and as maps, commuting triangles.

Conversely, each object A of a topos $\mathcal{C}$ gives rise to a space:

Definition

The *locale of subobjects* of $A \in \mathcal{C}$ is the partially ordered set

$$\mathcal{O}(A) = \{U \rightarrowtail A\}/\sim \ ;$$

this is a complete lattice satisfying (*), hence a locale.

Definition

A monoid M is an *inverse monoid* if for each $m \in M$ there is a unique m^* such that $mm^*m = m$ and $m^*mm^* = m^*$.

Such an M has a partial order $\leqslant$ and a *compatibility* relation $\smile$:

$$m \leqslant n \text{ iff } m = nm^*m$$
$$m \smile n \text{ iff } mn^*, m^*n \text{ idempotent}.$$

M is a *pseudogroup* if every pairwise compatible family has a join with respect to the partial order, and each $(-)m$ preserves joins.

Pseudogroups correspond to (localic) étale groupoids $G_1 \rightrightarrows G_0$.

Definition

Let M be a pseudogroup. An *M-set* is a set X with a right M-action and a map $(\bar{-})\colon X \to E(M)$ to the set of idempotents in M satisfying

$$x\bar{x} = x \qquad \text{and} \qquad \overline{xm} = m^*\bar{x}m\ .$$

Such an X also has a partial order and $\smile$ relation:

$$x \leqslant y \text{ iff } x = y\bar{x} \qquad \text{and} \qquad x \smile y \text{ iff } x\bar{y} = y\bar{x}\ .$$

X is an *M-sheaf* if every compatible family has a join which is preserved by the M-action.

If M corresponds to $\mathcal{G} = G_1 \rightrightarrows G_0$, then M-sheaves correspond to *equivariant sheaves* on $\mathcal{G}$: that is, local homeomorphisms $X \to G_0$ with a right action $X \times_{G_0} G_1 \to X$ over G_0.

Toposes vs pseudogroups

Any pseudogroup M gives rise to a topos:

Definition

The *topos of sheaves* on M is the category of M-sheaves and maps preserving the right action and $\overline{(-)}$.

Conversely, each object A of a topos $\mathcal{C}$ gives rise to a pseudogroup:

Definition

The *pseudogroup of partial isomorphisms* of $A \in \mathcal{C}$ is the set

$$\mathrm{PAut}(A) = \left\{ \begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ A & & A \end{array} \right\} / \sim \quad ,$$

with monoid structure given by composition of partial maps.

The Jonsson–Tarski topos

Definition

A *Jonsson–Tarski algebra* is a set X equipped with an isomorphism $X \to X \times X$.

Jonsson–Tarski algebras are models for an equational theory. It has two unary operations and one binary operation:

$$x \mapsto x\ell \qquad x \mapsto xr \qquad x, y \mapsto x * y$$

subject to the equations

$$(x * y)\ell = x \qquad (x * y)r = y \qquad (x\ell) * (xr) = x\,.$$

The Jonsson–Tarski topos

Proposition (Freyd, 1970s)

The category JT *of Jonsson–Tarski algebras is a topos.*

So associated to each object of JT is a group, a space and a pseudogroup. In fact, there's a very natural choice of object.

Definition

The Jonsson–Tarski algebra $F1$ is the free Jonsson–Tarski algebra on one generator.

The Jonsson–Tarski algebra

It's easy to describe $F1$ explicitly: its elements are binary trees

with leaves labelled in $\{\ell, r\}^*$, identified under the congruence

We can also write such trees in "two-line notation":

$$\begin{pmatrix} \ell\ell\ell & \ell\ell r & \ell r & r \\ \ell r\ell\ell & r\ell\ell & \varepsilon & \ell\ell \end{pmatrix}$$

where the first row is a maximal antichain for the prefix order.

Each element $h = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$ of $F1$ induces a local homeomorphism φ_h of Cantor space $\{\ell, r\}^\omega$ with $\varphi_h(x_i W) = y_i W$. Thus:

Theorem (Higman)

Aut($F1$) *is Thompson's group V.*

Proof.

By freeness, JT-maps $\gamma\colon F1 \to F1$ correspond to elements $h \in F1$ and so to functions $\varphi_h\colon \{\ell, r\}^\omega \to \{\ell, r\}^\omega$. Clearly γ is invertible just when φ_h is; and the group of invertible φ_h's is precisely V. □

The space associated to $F1$

Theorem

The locale $\mathcal{O}(F1)$ is the locale of opens of Cantor space.

Proof.

Call a right ideal $I \leqslant \{\ell, r\}^*$ *constructible* if $x\ell, xr \in I \Rightarrow x \in I$.

A constructible right ideal I corresponds to a sub-JT-algebra of $F1$: namely, that containing all trees with leaves from I.

It also corresponds to an open of Cantor space: namely, that containing all infinite extensions of words in I. □

The pseudogroup associated to $F1$

Theorem

The pseudogroup $\mathrm{PAut}(F1)$ *is the Cuntz pseudogroup* S_2*, i.e., the free pseudogroup generated by elements* ℓ, r *satisfying*

$$\ell^*\ell = r^*r = 1 \qquad \ell^*r = r^*\ell = \bot \qquad \ell\ell^* \vee rr^* = 1\,.$$

Proof.

Any $h \in \mathrm{PAut}(F1)$ can be written as $h = \left(\begin{smallmatrix} x_1 & x_2 & \cdots \\ y_1 & y_2 & \cdots \end{smallmatrix}\right)$ where both rows are antichains in $\{\ell, r\}^*$. Such an h corresponds

$$y_1x_1^* \vee y_2x_2^* \vee \ldots \qquad \in S_2\,.$$

□

The pseudogroup associated to $F1$

In fact, due to an observation of Freyd, JT is equivalent to the category of sheaves on $\mathrm{PAut}(F1) = S_2$. So JT is really the topos-theoretic incarnation of S_2.

Of course, we can translate this into the language of étale groupoids. The groupoid associated to S_2 is the Cuntz groupoid $\mathcal{G}_2$, whose object-space is Cantor space 2^ω, and whose arrows $V \to W$ are integers i such that eventually $V_n = W_{n-i}$.

So JT-algebras can also be identified with equivariant sheaves on $\mathcal{G}_2$. These are equally well sheaves on Cantor space $A \to 2^\omega$ endowed with a homeomorphism $A \cong A + A$ lifting $2^\omega \cong 2^\omega + 2^\omega$.

Topology and algebra in JT

Proposition

A space internal to JT *is a real-world space* A *endowed with an isomorphism* $A \to A + A$*.*

Proposition

Let k *be a (real-world) commutative ring. A* k*-module internal to* JT *is a real-world* k*-module* A *with an isomorphism* $A \to A \oplus A$*.*

Proposition

The ring of k*-linear endomorphisms of the free* k*-module* $k(F1)$ *in* JT *is the Leavitt algebra* $L_{2,k}$*.*

Danger! I am not an analyst.

Proposition

The ring of complex numbers in JT is the real-world ring $C(2^\omega)$ *with self-similarity induced from* $2^\omega \cong 2^\omega + 2^\omega$.

Proposition

A Hilbert space internal to JT is a real-world Hilbert $C(2^\omega)$*-module* H *endowed with a linear isomorphism* $H \to H \oplus H$ *compatible with the self-similarity of* $C(2^\omega)$.

Question: is this the same as a Hilbert $\mathcal{O}_2$-module?

Proposition

The ring of endomorphisms of the free complex vector space $\mathbb{C}(F1)$ in JT is the convolution algebra $C_c(\mathcal{G}_2)$.

Proposition

$F1$ has decidable equality, and so $\mathbb{C}(F1)$ is an inner product space in JT. Its norm-completion $\ell^2(F1)$ is the real-world Hilbert $C(2^\omega)$-module completion of $C_c(\mathcal{G}_2)$.

I believe that, in fact:

Proposition

The C^*-algebra of bounded linear operators on $\ell^2(F1)$ in JT is the reduced C^*-algebra of $\mathcal{G}_2$.

Generalisations

Generalised self-similarity

We have seen that JT encodes the circle of structures associated to the Cuntz C^*-algebra $\mathcal{O}_2$.

In 2007, Tom Leinster described a general way of building "self-similar toposes" like JT.

Using this, one can exhibit other self-similar toposes which encode structures linked to, say, directed (higher) graphs or self-similar groups and groupoids—but also more exotic things.

Directed graphs

A *directed graph* $G = (V, E)$ comprises a set of vertices V and an indexed family of sets of edges $\mathrm{E} = (E_{vw})_{v,w \in V}$.

Given a vector of sets $(X_v)_{v \in V}$ we can define a new vector X^G by

$$(X^G)_v = \prod_{w \in V} (X_w)^{E_{vw}} .$$

Definition

A *Jonsson–Tarski graph algebra* over G is a vector of sets $(X_v)_{v \in V}$ equipped with an isomorphism $X \to X^G$.

Proposition (Leinster)

The category JT_G *of Jonsson–Tarski graph algebras is a topos.*

So we can play the same game as before.

Directed graph toposes

Definition

The Jonsson–Tarski graph algebra $F1$ is the free Jonsson–Tarski algebra on the constant vector $1 = (1, 1, \dots)$.

We find, for example:

- The space $\mathcal{O}(F1)$ is the path space $P(G)$;
- The pseudogroup $\mathrm{PAut}(F1)$ is the Cuntz–Kreiger pseudogroup;
- The associated étale groupoid is the graph groupoid $\mathcal{G}$.
- JT_G is the category of equivariant sheaves on $\mathcal{G}$.
- The ring of linear endos of $k(F1)$ is the Leavitt path k-algebra;
- (Hopefully) $B(\ell^2(F1)) = C^*_{\mathrm{red}}(\mathcal{G})$.

Self-similar group actions

A *self-similar action* of a group G on a set A is a function

$$\delta \colon G \times A \to A \times G \qquad (g, a) \mapsto (g \cdot a,\ g|_a) \,.$$

satisfying associativity and unit laws. For any right G-set X, there is a right G-set structure on X^A given by

$$(\varphi \cdot g)(a) = \varphi(g \cdot a) \cdot g|_a \,.$$

Definition

A *Nekrashevych algebra* over δ is a right G-set X equipped with an equivariant isomorphism $X \to X^A$.

Proposition

The category JT_δ *of Nekrashevych algebras is a topos.*

And we can calculate away as before.

Self-similar directed graphs

If $G = (E, V)$ is a directed graph, define $G^2 \coloneqq (E^2, V)$ where

$$(E^2)_{uw} = \textstyle\sum_{v \in V} E_{uv} \times E_{vw} \ .$$

Definition

A *Leinster algebra* is a directed graph $G = (E, V)$ together with an identity-on-vertices isomorphism $G \to G^2$.

Proposition (Leinster)

The category JT_L *of Leinster algebras is a topos.*

The calculations in this case are interesting!

Definition

The Leinster algebra FE is the free Leinster algebra on an edge.

Proposition

$\mathrm{Aut}(FE)$ *is Thompson's group F.*

Proposition

The pseudogroup $\mathrm{PAut}(FE)$ *is the free pseudogroup generated by elements* ℓ, r *satisfying*

$$\ell^*\ell = r^*r = 1 \qquad \ell\ell^* \vee rr^* = 1$$
$$\ell^*rr = r^*\ell\ell = \bot \qquad \ell^*r\ell = \ell^*r \qquad r^*\ell r = r^*\ell \qquad .$$

The space $\mathcal{O}(FE)$ turns out to be the following *combinatorial interval*. Write $\mathbb{Q}_2$ for the set of dyadic rationals.

Definition

The *combinatorial interval* $\mathbb{I}$ is the set

$$[0,1] \cup \{x^+ : x \in \mathbb{Q}_2 \cap (0,1]\} \cup \{x^- : x \in \mathbb{Q}_2 \cap [0,1)\}$$

topologised with basic open sets

$$[a,b] \cup \{x^+ : x \in \mathbb{Q}_2 \cap (a,b]\} \cup \{x^- : x \in \mathbb{Q}_2 \cap [a,b)\}$$

for $a, b \in \mathbb{Q}_2 \cap [0,1]$.

I think $\mathbb{I}$ is the primitive spectrum of the Farey AF algebra. So is the C^*-algebra of $\mathrm{PAut}(FE)$ related to the Farey AF algebra?