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VAN DER POL EXPANSIONS OF L-SERIES

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Abstract. We provide concise series representations for various L-series integrals. Different techniques are needed below and above the abscissa of absolute convergence of the underlying L-series.

1. Preliminaries. In [8] the following odd looking integral evaluation is obtained.

$$\int_0^\infty \frac{\left(3-2\sqrt{2}\cos\left(t\log 2\right)\right)\,\left|\zeta\left(1/2+it\right)\right|^2}{t^2+1/4}\,dt=\pi\log 2. \tag{1}$$

This identity turns out—formally—to be a case of a rather pretty, and perhaps useful, class of L-series evaluations given in Theorem 1 and Corollary 1 (*cf.* [2]). In Theorem 3 we recover (1) entirely rigorously.

Given a *Dirichlet series*

$$\lambda(s):=\sum_{n=1}^\infty \frac{\lambda_n}{n^s},\; s=\sigma+i\tau,\; \sigma=\Re s>0,$$

we consider the integral

$$\iota_\lambda(\sigma):=\frac{1}{2}\int_{-\infty}^\infty \left|\frac{\lambda(s)}{s}\right|^2 d\tau$$

as a function of λ. Observe that when the coefficients λ_n are real

$$\iota_\lambda(\sigma)=\int_0^\infty \left|\frac{\lambda(s)}{s}\right|^2 d\tau,$$

but that this is not necessarily so when the coefficients are complex. We refer to [3, 5, 7] for other, largely standard details.

2. Integrals Involving s with Large Real Part. It is convenient to recall [2] that, for $u, a>0$,

$$\int_{-\infty}^\infty \frac{\cos\left(at\right)}{t^2+u^2}dt=\frac{\pi}{u}\,e^{-au}. \tag{2}$$

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Theorem 1. *For $a(s) := \sum_{n=1}^\infty a_n n^{-s}$, $b(s) := \sum_{n=1}^\infty b_n n^{-s}$, and $s = \sigma + i\,\tau$ with fixed $\sigma = \Re(s) > 0$ such that both Dirichlet series are absolutely convergent, we have*

$$\iota_{a,b}(\sigma) := \frac{1}{2}\int_{-\infty}^{\infty} \frac{a(s)\,\overline{b}(s)}{\sigma^2+\tau^2}\,d\tau = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} \frac{A_n\,\overline{B}_n - A_{n-1}\,\overline{B}_{n-1}}{n^{2\sigma}}, \tag{3}$$

where $A_n := \sum_{k=1}^n a_k,\ B_n := \sum_{k=1}^n b_k,\ A_0 := B_0 := 0.$

Proof. Let $a_N(s) := \sum_{n=1}^N a_n n^{-s}, \quad b_N(s) := \sum_{n=1}^N b_n n^{-s}$. Then, in view of (2), we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{a_N(s)\overline{b}_N(s)}{\sigma^2+\tau^2}\,d\tau &= \int_{-\infty}^{\infty} \frac{\sum_{N\ge n,m>0} a_n\,\overline{b}_m n^{-\sigma+i\tau} m^{-\sigma-i\tau}}{\sigma^2+\tau^2}\,d\tau \\
&= \sum_{N\ge n>m>0} \frac{a_n\overline{b}_m}{(nm)^\sigma}\int_{-\infty}^{\infty} \frac{\cos\big(\tau\log(n/m)\big)}{\sigma^2+\tau^2}\,d\tau \\
&+ \sum_{N\ge n>m>0} \frac{a_m\overline{b}_n}{(nm)^\sigma}\int_{-\infty}^{\infty} \frac{\cos\big(\tau\log(m/n)\big)}{\sigma^2+\tau^2}\,d\tau + \sum_{n=1}^{N}\frac{a_n\overline{b}_n}{n^{2\sigma}}\int_{-\infty}^{\infty}\frac{1}{\sigma^2+\tau^2}\,d\tau \\
&\qquad + i\sum_{N\ge m,n>0} \frac{a_n\overline{b}_m}{(nm)^\sigma}\int_{-\infty}^{\infty} \frac{\sin\big(\tau\log(n/m)\big)}{\sigma^2+\tau^2}\,d\tau \\
&= \sum_{N\ge n>m>0} \frac{a_n\overline{b}_m + a_m\overline{b}_n}{(nm)^\sigma}\int_{-\infty}^{\infty} \frac{\cos\big(\tau\log(n/m)\big)}{\sigma^2+\tau^2}\,d\tau + \frac{\pi}{\sigma}\sum_{n=1}^{N}\frac{a_n\overline{b}_n}{n^{2\,\sigma}} \\
&= \frac{\pi}{\sigma}\sum_{N\ge n>m>0} \frac{a_n\overline{b}_m + a_m\overline{b}_n}{(nm)^\sigma (n/m)^\sigma} + \frac{\pi}{\sigma}\sum_{n=1}^{N}\frac{a_n\overline{b}_n}{n^{2\,\sigma}} \\
&= \frac{\pi}{\sigma}\sum_{n=1}^{N}\frac{a_n\overline{B}_{n-1} + A_{n-1}\overline{b}_n + a_n\overline{b}_n}{n^{2\sigma}} = \frac{\pi}{\sigma}\sum_{n=1}^{N}\frac{A_n\,\overline{B}_n - A_{n-1}\,\overline{B}_{n-1}}{n^{2\sigma}}.
\end{aligned}$$

Note that the imaginary part in the above evaluation vanished because we integrated an odd function over the range $-\infty < \tau < \infty$.

Next, we observe that

$$|a_N(s)\overline{b}_N(s)| \le \sum_{n=1}^{\infty}\frac{|a_n\overline{b}_n|}{n^{2\sigma}} = M < \infty,$$

where M is independent of τ. Hence

$$\left|\frac{a_N(s)\overline{b}_N(s}{\sigma^2+\tau^2}\right| \le \frac{M}{\sigma^2+\tau^2},$$

and (3) follows by Lebesgue's theorem on dominated convergence on letting $N \to \infty$. □

As an immediate consequence we have:

Corollary 1. *If* $\lambda(s) := \sum_{n=1}^\infty \lambda_n n^{-s}$ *with* $s = \sigma + i\,\tau$ *and fixed* $\sigma = \Re(s) > 0$ *such that the Dirichlet series is absolutely convergent, then*

$$\iota_\lambda(\sigma) := \frac{1}{2}\int_{-\infty}^{\infty} \left|\frac{\lambda(s)}{s}\right|^2 d\tau = \frac{\pi}{2\sigma}\sum_{n=1}^\infty \frac{|\Lambda_n|^2 - |\Lambda_{n-1}|^2}{n^{2\sigma}}, \tag{4}$$

where $\Lambda_n := \sum_{k=1}^n \lambda_k$, $\Lambda_0 := 0$.

If, in addition, all the coefficients λ_n *are real, then*

$$\iota_\lambda(\sigma) = \int_0^{\infty} \left|\frac{\lambda(s)}{s}\right|^2 d\tau = \frac{\pi}{2\sigma}\sum_{n=1}^\infty \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}}. \tag{5}$$

Note that, by Dirichlet's test [9], the final series in (5) is convergent for all $\sigma > 0$ when Λ_n is bounded, but we cannot automatically guarantee that it is equal to the integral in in this case, or even that the integral is finite. Simple continuation arguments will not work. Of course, similar difficulties arise with regard to (3) and (4). This in part motivates the first example and the following section.

It is, however, easy now to check that

$$\langle a, b\rangle_\sigma := \frac{1}{2}\int_{-\infty}^{\infty} \frac{a(s)\,\overline{b}(s)}{\sigma^2+\tau^2}\,d\tau$$

defines an extended-value inner product on the space of Dirichlet series with $\langle \alpha, \alpha\rangle_\sigma = \iota_\alpha(\sigma)$, which is typically finite for σ large enough.

In the sequel, we let $\mathrm{L}_\mu(s) := \sum_{n=1}^\infty (\frac{\mu}{n}) n^{-s}$ denote the *primitive L-function* corresponding to the *Kronecker symbol* $(\frac{\mu}{n})$, [3]. Below, $\lfloor x\rfloor$ is the integer part and $\lceil x\rfloor$ is the truncation of x, so that

$$\lceil x\rfloor = \begin{cases} \lfloor x\rfloor & \text{when } x \ge 0,\\ -\lfloor x\rfloor & \text{when } x < 0.\end{cases}$$

As usual, $\{x\} := x - \lfloor x\rfloor$ denotes the fractional part of x.

Example 1. *For the* Riemann zeta function (ζ=L_1)*, and for* $\sigma > 1$, *Corollary* 1 *applies and yields*

$$\frac{\sigma}{\pi}\,\iota_\zeta(\sigma) = \zeta(2\sigma-1) - \frac{1}{2}\,\zeta(2\sigma),$$

as $\lambda_n = 1$ *and* $\Lambda_n = n$. *By contrast it is known—see equation* (69) *of* [6]*—that on the critical line* $s = \frac{1}{2} + i\tau$

$$\frac{1}{2\,\pi}\,\iota_\zeta\left(\frac{1}{2}\right) = \log(\sqrt{2\,\pi}) - \frac{1}{2}\,\gamma.$$

More broadly, for $0 < \sigma < 1$, Crandall (recorded in [6]) has found that

$$\iota_\zeta(\sigma) = \pi\int_0^1 \sum_{n=0}^\infty \frac{\theta^2}{(n+\theta)^{1+2\sigma}}\,d\theta = \pi\int_0^1 \theta^2\zeta(1+2\sigma,\theta)\,d\theta, \tag{6}$$

where

$$\zeta(s,a) := \sum_{n=0}^{\infty} (n+a)^{-s}$$

is the Hurwitz zeta function—which is easy to compute. This devolves from *van der Pol's representation*

$$\frac{\zeta(s)}{s} = -\int_{-\infty}^{\infty} e^{-\sigma\omega}\left(e^{\omega} - \lfloor e^{\omega} \rfloor\right) e^{-i\tau\omega}\, d\omega, \quad s = \sigma + i\tau \text{ with } 0 < \sigma < 1. \tag{7}$$

One way to obtain identity (7) is to note that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-s\omega}(e^{\omega} - \lfloor e^{\omega} \rfloor)\, d\omega &= \int_{0}^{\infty} t^{-s-1}(t - \lfloor t \rfloor)\, dt = \sum_{n=0}^{\infty} \int_{n}^{n+1} t^{-s-1}(t - \lfloor t \rfloor)\, dt \\ &= \int_0^1 \theta\, \zeta(1+s,\theta)\, d\theta = \sum_{n=0}^{\infty} \int_0^1 \theta\, (\theta+n)^{-s-1}\, d\theta \\ &= -\frac{1}{s} \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} \right)\end{aligned}$$

on integrating once by parts. Now set

$$\sigma_N := \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s}$$

and observe that we have shown that σ_N converges to some number σ_∞. Further

$$\sigma_{2N} - 2^{1-s}\, \sigma_N = \sum_{n=1}^{2N} \frac{1}{n^s} - 2\sum_{n=1}^{N} \frac{1}{(2n)^s} = \sum_{n=1}^{2N} \frac{(-1)^{n+1}}{n^s} \to \left(1 - 2^{1-s}\right)\zeta(s),$$

which implies that $\sigma_\infty = \zeta(s)$, as required.

Another way to establish (7) (kindly suggested by the David Bradley) is to start with the standard identity

$$\frac{\zeta(s)}{s} = \frac{1}{s-1} - \int_1^{\infty} x^{s-1}(x - \lfloor x \rfloor)\, dx$$

which can easily be derived under the restriction $\Re s > 1$ and then extended by analytic continuation to the punctured half-plane $\{s\,, \Re s > 0,\ s \neq 1\}$. This together with the identitity

$$\frac{1}{s-1} = -\int_0^1 x^{s-1}(x - \lfloor x \rfloor)\, dx$$

which is valid for $\Re s < 1$ yields (7) via the change of variable $x = e^{\omega}$.

We now write (7) as a Fourier transform:

$$\frac{\zeta(s)}{s} = -\mathcal{F}_\tau\left(e^{-\sigma\omega}\{e^{\omega}\}\right)$$

and so obtain, for $0 < \sigma < 1$,

$$2\,\iota_\zeta(\sigma) = \int_{-\infty}^{\infty} \left|\frac{\zeta(s)}{s}\right|^2 d\tau = 2\,\pi \int_{-\infty}^{\infty} e^{-2\sigma\omega} |e^\omega - \lfloor e^\omega \rfloor|^2\, d\omega$$

from the $\mathcal{L}_2$ Plancherel theorem [9].

It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\sigma\omega}|e^\omega - \lfloor e^\omega\rfloor|^2\, d\omega &= \int_0^\infty t^{-2\sigma-1}|t-\lfloor t\rfloor|^2\, dt \\ &= \sum_{n=0}^{\infty}\int_n^{n+1} t^{-2\sigma-1}|t-\lfloor t\rfloor|^2\, dt = \sum_{n=0}^{\infty}\int_0^1 \theta^2\,(\theta+n)^{-2\sigma-1}\, d\theta \\ &= \int_0^1 \theta^2 \sum_{n=0}^{\infty}(\theta+n)^{-2\sigma-1}\, d\theta,\end{aligned}$$

as required to yield (6). Note that all terms are absolutely convergent which legitimates the operations. We have also established, inter alia, that, for $0 < \sigma < 1$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_0^1 \theta\,\zeta(1+s,\theta)\, d\theta\right|^2 d\tau = \int_0^1 \theta^2\,\zeta(1+2\sigma,\theta)\, d\theta.$$

Moreover, reversing the order of integration and summation above leads to

$$\iota_\zeta(\sigma) = \pi\sum_{n=0}^{\infty}\int_0^1 \frac{\theta^2}{(n+\theta)^{1+2\sigma}}\, d\theta = -\frac{\pi}{2\sigma}\left(\frac{2\,\zeta(2\sigma-1)}{2\sigma-1} + \zeta(2\sigma)\right)$$

which in the limit as $\sigma \to \frac{1}{2}$ recaptures the evaluation quoted above. Recapitulating, we have

$$\frac{\sigma}{\pi}\,\iota_\zeta(\sigma) = \begin{cases} -\dfrac{1}{2}\,\zeta(2\sigma) - \dfrac{\zeta(2\sigma-1)}{2\sigma-1}, & 0<\sigma<1; \\ -\dfrac{1}{2}\,\zeta(2\sigma) + \zeta(2\sigma-1), & 1<\sigma<\infty. \end{cases} \tag{8}$$

There are similar formulae for $s \mapsto \zeta(s-k)$ with k integral. For instance, applying the result in (5) with $\zeta_1 := t \mapsto \zeta(t+1)$ yields

$$\frac{1}{\pi}\int_0^\infty \frac{|\zeta(3/2+i\tau)|^2}{1/4+\tau^2}\, d\tau = \frac{1}{\pi}\,\iota_{\zeta_1}\left(\frac{1}{2}\right) = 2\,\zeta(2,1) + \zeta(3) = 3\,\zeta(3),$$

on using Euler's result (see [4]) that $\zeta(2,1) := \sum_{n=2}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{k} = \zeta(3)$. □

Example 2. *For the* alternating zeta function, $\alpha := s \mapsto (1-2^{1-s})\zeta(s)$, *we recover, as in* [7], *that*

$$\frac{\sigma}{\pi}\,\iota_\alpha(\sigma) = \frac{1}{2}\,\alpha(2\sigma),$$

as $\lambda_n = (-1)^{n+1}$, $\Lambda_n = (1-(-1)^n)/2$ *and* $\Lambda_n^2 - \Lambda_{n-1}^2 = (-1)^{n+1}/2$.

Set $\sigma := 1/2$. Then

$$\frac{|\alpha(1/2+it)|^2}{1/4+t^2} = \frac{|1-2^{1/2-i\,t}|^2\,|\zeta(1/2+i\,t)|^2}{1/4+t^2}$$
$$= \left(3-2\sqrt{2}\cos\left(t\ln\left(2\right)\right)\right)\frac{|\zeta(1/2+i\,t)|^2}{1/4+t^2}$$

is precisely the integrand in (1). Thus, since $\alpha(2\sigma) = \log 2$ we see that $\iota_\alpha(1/2) = \pi\,\log 2$, is the evaluation in [7]. Note that to justify the exchange of sum and integral implicit in (5) we should have to more carefully analyse the integrand, since 1/2 is below the abscissa of absolute convergence of the series, and note that this would not have been legitimate in Example 1 because of (8). This motivates the approach in the section below on the Hurwitz zeta function. □

Example 3. (a) *For the* Catalan zeta function (β=L_{-4})*, and for* $\sigma > 1$:

$$\frac{\sigma}{\pi}\,\iota_\beta(\sigma) = \frac{1}{2}\,\beta(2\sigma),$$

as $\lambda_{2n} = 0, \lambda_{2n+1} = (-1)^n$ *and again* $\Lambda_n^2 - \Lambda_{n-1}^2 = \lambda_n$.

(b) *For* L_{-3}*, the same pattern holds, in that* $\frac{\sigma}{\pi}\,\iota_{\mathrm{L}_{-3}}(\sigma) = \frac{1}{2}\,\mathrm{L}_{-3}(2\sigma)$*, but not for* L_5, L_{-7}*, and so on.*

(c) *In general the series* $\mathrm{L}_{\pm d}$ *does not lead to output which is again a primitive L-series modulo* d.

For example,

$$\frac{\sigma}{\pi}\,\iota_{\mathrm{L}_5}(\sigma) = -\sum_{5\nmid n}\frac{(-1)^{n \mod 5}}{n^{2\sigma}}, \text{ and } \frac{\sigma}{\pi}\,\iota_{\mathrm{L}_{-8}}(\sigma) = \mathrm{L}_{-8}(2\sigma) - \frac{1}{2}\,\mathrm{L}_{-4}(2\sigma).$$

These are not character sums, though always the coefficients repeat modulo d.

(d) *Finally, let* $\vartheta := s \mapsto \left(1+2^{-s}\right)^{-1}$*. Then again* $\frac{\sigma}{\pi}\,\iota_\vartheta(\sigma) = \frac{1}{2}\,\vartheta(2\sigma)$. □

For each of these examples, which are discussed further in [2], the defect of Theorem 1 and Corollary 1 is that, as we have seen, they only directly apply when σ is large enough. The van der Pol approach offers a nice alternative, especially in the critical strip.

3. Van der Pol's Approach to Hurwitz Zeta. Recall that

$$\zeta(s,a) := \sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\ a > 0,$$

is initially defined for $\Re s > 1$ and appropriately analytically continued for $\Re s < 1$. Thus, $\zeta(s) = \zeta(s,1)$. We first work out a Fourier transform representation for $\zeta(a,s)$.

Proposition 1. *For* $1 \ge a > 0$ *and* $1 > \Re s > 0$*, we have*

$$\int_{-\infty}^{\infty} e^{-s\omega}\left(e^\omega - \lceil e^\omega - a\rfloor\right)d\omega = -\frac{1}{s}\,\zeta(s,a+1). \tag{9}$$

Proof. (i) Observe that the formula uses the truncation of $e^w - a$ and not the floor, and that the two only differ in the interval $-\infty < w < \log a$. We have

$$\begin{aligned}\int_{\log a}^{\infty} e^{-s\omega}\{e^{\omega}-a\}\,d\omega &= \int_{a}^{\infty} t^{-s-1}(t-a-\lfloor t-a\rfloor)\,dt\\ &= \sum_{n=0}^{\infty}\int_{n+a}^{n+1+a} t^{-s-1}(t-a-\lfloor t-a\rfloor)\,dt\\ &= \int_0^1 \theta\,\zeta(1+s,\theta+a)\,d\theta = \sum_{n=0}^{\infty}\int_0^1 \theta\,(\theta+n+a)^{-s-1}\,d\theta \qquad (10)\\ &= -\frac{1}{s}\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{(n+a)^s}-\frac{(N+a)^{1-s}-a^{1-s}}{1-s}\right)\\ &= -\frac{1}{s}\,\zeta(s,a+1)-\frac{a^{1-s}}{s(1-s)},\end{aligned}$$

this evidently being true for $\Re s > 1$ and so for $0 < \Re s < 1$ by analytic continuation– both sides of (10) clearly being analytic for $\Re s > 0$. Incidentally, we have shown that (10) is valid for $\Re s > 0,\ a > 0$. The additional restriction $\Re s < 1,\ a \leq 1$ is needed for the next part of the proof.

It follows from (10) that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-s\omega}(e^{\omega}-\lceil e^{\omega}-a\rfloor)\,d\omega &= \int_{\log a}^{\infty} e^{-s\omega}\left(\{e^{\omega}-a\}+a\right)\,d\omega + \int_{-\infty}^{\log a} e^{(1-s)\omega}\,d\omega\\ &= -\frac{1}{s}\,\zeta(s,a+1)-\frac{a^{1-s}}{s(1-s)}+\frac{a^{1-s}}{s}+\frac{a^{1-s}}{1-s} = -\frac{1}{s}\,\zeta(s,a+1),\end{aligned}$$

and this establishes (9). □

(ii) Another proof of equation (10) (suggested by the David Bradley) is to proceed from Theorem 12.2 in Apostol [1, p. 269] which states that for $0 < a \leq 1$, $\Re s > 0$, $N = 0, 1, 2 \ldots$,

$$\zeta(s,a) := \sum_{n=0}^{N}\frac{1}{(n+a)^s} - \frac{(N+a)^{1-s}}{s-1} - s\int_N^{\infty}\frac{\{x\}}{(x+a)^{s+1}}\,dx. \qquad (11)$$

Putting $N = 0$ in (11) and making the change of variable $x = e^{\omega} - a$ gives equation (10). □

Proposition 2. *For $\Re s > 0$ and $a \geq 0$, we have*

$$\sigma(s,a) := \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{(n+a)^s}-\frac{(N+a)^{1-s}}{1-s}\right) = \zeta(s,a+1).$$

Another form of the limit is

$$\sigma(s,a) = -s\int_0^1 t\zeta(1+s,a+t)\,dt,$$

and indeed $\sigma(s, 0) = \zeta(s)$.

Proof. The case $a > 0$ follows immediately from the proof of (10) in part (i) above, and the case $a = 0$ was treated in the earlier proof of (7). We can also derive the case $1 \geq a > 0$ by letting $N \to \infty$ in (11) and observing that $\zeta(s, a) - a^{-s} = \zeta(s, a+1)$. □

Most of what follows concerns Dirichlet series having coefficients which repeat modulo N.

Theorem 2. *Let* $\lambda(s) := \sum_{n=0}^{\infty} \lambda_n n^{-s}$, $s = \sigma + i\tau$, *with coefficients,* λ_n, *repeating modulo* N *(i.e.,* $\lambda_{N+n} = \lambda_n$*). Then, for* $0 < \sigma < 1$,

$$- \int_{-\infty}^{\infty} e^{-\sigma\omega} \sum_{k=1}^{N} \lambda_k \left(\frac{e^\omega}{N} - \left\lceil \frac{e^\omega + N - k}{N} \right\rceil \right) e^{-i\tau\omega}\, d\omega = \frac{\lambda(s)}{s}. \tag{12}$$

Proof. Observe first that

$$\lambda(s) = \sum_{m=0}^{\infty} \sum_{k=1}^{N} \frac{\lambda_k}{(mN+k)^s} = \frac{1}{N^s} \sum_{k=1}^{N} \lambda_k\, \zeta\left(s, \frac{k}{N}\right); \tag{13}$$

strictly this is true for $\sigma > 1$ in the first place and the equating of the extreme terms for $\sigma > 0$ follows by analytic continuation.

It follows from (9) that, for $k = 1, 2, \ldots, N$,

$$-\frac{1}{N^s} \int_{-\infty}^{\infty} e^{-\sigma\omega} \left(e^\omega - \left\lceil e^\omega - \frac{k}{N} \right\rceil \right) e^{-i\tau\omega}\, d\omega = \frac{1}{s\,N^s} \zeta\left(s, \frac{k}{N}\right) - \frac{1}{s\,k^s}.$$

We now change variables—$w \mapsto \omega - \log N$—to obtain

$$- \int_{-\infty}^{\infty} e^{-\sigma\omega} \left(\frac{e^\omega}{N} - \left\lceil \frac{e^\omega - k}{N} \right\rceil \right) e^{-i\tau\omega}\, d\omega = \frac{1}{s\,N^s} \zeta\left(s, \frac{k}{N}\right) - \frac{1}{s\,k^s},$$

so that

$$- \int_{-\infty}^{\infty} e^{-\sigma\omega} \left(\frac{e^\omega}{N} - \left\lceil \frac{e^\omega + N - k}{N} \right\rceil \right) e^{-i\tau\omega}\, d\omega = \frac{1}{s\,N^s} \zeta\left(s, \frac{k}{N}\right) \tag{14}$$

since

$$\left\lceil \frac{e^\omega + N - k}{N} \right\rceil - \left\lceil \frac{e^\omega - k}{N} \right\rceil = \begin{cases} 1 & \text{when } w \geq \log k, \\ 0 & \text{when } w < \log k. \end{cases}$$

Summing (14) for $k = 1, 2, \ldots, N$, and applying (13) yields (12), as desired. □

We are now in position to prove the following companion to Theorem 1 in which we use the notation: given a Dirichlet series $\lambda(s) := \sum_{n=1}^{\infty} \lambda_n n^{-s}$ with coefficients repeating modulo N, we define an associated *kernel*

$$W_N(\lambda, t) := \sum_{k=1}^{N} \lambda_k \left\lceil t + \frac{N-k}{N} \right\rceil.$$

Theorem 3. *Suppose that the coefficients of the two Dirichlet series* $a(s) := \sum_{n=1}^\infty a_n n^{-s}$, $b(s) := \sum_{n=1}^\infty b_n n^{-s}$ *repeat modulo* N, *that* $A_n := \sum_{k=1}^n a_k$, $B_n := \sum_{k=1}^n b_k$, $A_0 := B_0 := 0$, *and that*

$$\iota_{a,b}(\sigma) := \frac{1}{2}\int_{-\infty}^{\infty} \frac{a(s)\,\overline{b}(s)}{\sigma^2+\tau^2}\,d\tau \text{ with } s = \sigma + i\tau.$$

Suppose further that $A_N = B_N = 0$. *Then, for* $0 < \sigma < 1$,

$$\iota_{a,b}(\sigma) = \frac{\pi}{N^{2\sigma}} \int_0^1 \zeta(2\sigma+1,t)\,W_N(a,t)\,W_N(\overline{b},t)\,dt. \tag{15}$$

Moreover, for all $\sigma > 0$,

$$\iota_{a,b}(\sigma) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} \frac{A_n\,\overline{B}_n - A_{n-1}\,\overline{B}_{n-1}}{n^{2\sigma}}. \tag{16}$$

Proof. Suppose first that $1 > \sigma > 0$. Since $A_N = B_N = 0$, it follows from Theorem 2 that

$$\frac{a(s)}{s} = \int_{-\infty}^{\infty} e^{-\sigma\omega} \sum_{k=1}^{N} a_k \left\lceil \frac{e^\omega + N - k}{N} \right\rceil e^{-i\tau\omega}\,d\omega,$$

with a corresponding formula for $b(s)/s$. Hence, by the $\mathcal{L}_2$ Plancherel theorem [9],

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{a(s)\,\overline{b}(s)}{\sigma^2+\tau^2}\,d\tau \\
&= 2\pi \int_{-\infty}^{\infty} e^{-2\sigma\omega} \left(\sum_{k=1}^{N} a_k \left\lceil \frac{e^\omega + N - k}{N} \right\rceil\right) \left(\sum_{k=1}^{N} \overline{b}_k \left\lceil \frac{e^\omega + N - k}{N} \right\rceil\right) d\omega \\
&= 2\pi \sum_{n=0}^{N} \int_0^N \frac{1}{(Nn+u)^{2\sigma+1}} \left(\sum_{k=1}^{N} a_k \left\lceil \frac{u}{N} + \frac{N-k}{N} \right\rceil\right) \\
&\qquad\qquad \times \left(\sum_{k=1}^{N} \overline{b}_k \left\lceil \frac{u}{N} + \frac{N-k}{N} \right\rceil\right) du \qquad (17) \\
&= \frac{2\pi}{N^{2\sigma}} \int_0^1 \zeta(2\sigma+1,t) \left(\sum_{k=1}^{N} a_k \left\lceil t + \frac{N-k}{N} \right\rceil\right) \\
&\qquad\qquad \times \left(\sum_{k=1}^{N} \overline{b}_k \left\lceil t + \frac{N-k}{N} \right\rceil\right) dt \\
&= \frac{2\pi}{N^{2\sigma}} \int_0^1 \zeta(2\sigma+1,t)\,W_N(a,t)\,W_N(\overline{b},t)\,dt.
\end{aligned}$$

This establishes (15). We now denote the characteristic function of the interval $(k/N, (k+1)/N)$ by χ_k and observe that, since $A_N = \overline{B}_N = 0$, we can use summation by parts to re-express the kernels as follows:

$$W_N(a,t) = \sum_{k=1}^{N-1} A_k\,\chi_k(t), \quad W_N(\overline{b},t) = \sum_{k=1}^{N-1} \overline{B}_k\,\chi_k(t).$$

Thus,

$$W_N(a,t)W_N(\overline{b},t) = \sum_{k=1}^{N-1} A_k\overline{B}_k\,\chi_k(t).$$

Consequently, by (15),

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{a(s)\,\overline{b}(s)}{\sigma^2+\tau^2}\,d\tau &= \frac{2\pi}{N^{2\sigma}} \sum_{k=1}^{N-1} A_k\overline{B}_k \int_{\frac{k}{N}}^{\frac{k+1}{N}} \zeta(2\sigma+1,t)dt \\ &= \frac{\pi}{\sigma}\sum_{n=0}^{\infty}\sum_{k=1}^{N-1} A_k\overline{B}_k\left(\frac{1}{(Nn+k)^{2\sigma}} - \frac{1}{(Nn+k+1)^{2\sigma}}\right) \\ &= \frac{\pi}{\sigma}\sum_{n=0}^{\infty}\sum_{k=1}^{N} \frac{A_k\overline{B}_k - A_{k-1}\overline{B}_{k-1}}{(Nn+k)^{2\sigma}} = \frac{\pi}{\sigma}\sum_{m=1}^{\infty}\frac{A_m\overline{B}_m - A_{m-1}\overline{B}_{m-1}}{m^{2\sigma}},\end{aligned}$$

and this shows that (16) holds when $0<\sigma<1$.

By Theorem 1, (16) also holds when $\sigma>1$, since the Dirichlet series defining $a(s)$ and $b(s)$ are absolutely convergent in this range. Further, since $A_n\,\overline{B}_n - A_{n-1}\,\overline{B}_{n-1}$ is bounded, by Dirichlet's test, the series in (16) is absolutely convergent and thus continuous as a function of σ for $\sigma>0$. Finally, it is easy to show, by means of partial summation, that $a(s)$ and $b(s)$ are analytic in the disk $\{s\,, |s-1|<\frac{1}{4}\}$, and hence bounded therein by a constant M, say. It follows, by Lebesgue's theorem on dominated convergence, that $\iota_{a,b}(\sigma)$ is continuous for $\frac{3}{4}<\sigma<\frac{5}{4}$, and hence that (16) holds for all $\sigma>0$. □

As an immediate consequence we have the following companion to Corollary 1.

Corollary 2. *Suppose that the coefficients of the Dirichlet series* $\lambda(s) := \sum_{n=1}^{\infty}\lambda_n n^{-s}$ *repeat modulo* N, *and that* $\Lambda_n := \sum_{k=1}^{n}\lambda_k$, $\Lambda_0 := 0$. *Suppose further that* $\Lambda_N = 0$. *Then, for* $s=\sigma+i\tau$ *with* $\sigma>0$, *we have*

$$\iota_\lambda(\sigma) := \frac{1}{2}\int_{-\infty}^{\infty}\left|\frac{\lambda(s)}{s}\right|^2 d\tau = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty}\frac{|\Lambda_n|^2 - |\Lambda_{n-1}|^2}{n^{2\sigma}},$$

where $\Lambda_n := \sum_{k=1}^{n}\lambda_k$, $\Lambda_0 := 0$.
If, in addition, all the coefficients λ_n *are real, then*

$$\iota_\lambda(2\sigma) = \int_0^{\infty}\left|\frac{\lambda(s)}{s}\right|^2 d\tau = \frac{\pi}{\sigma}\sum_{n=1}^{\infty}\frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}}.$$

Note that for $N=2, \lambda_1 = 1 = -\lambda_2$ we reobtain from Corollary 2, a rigorous form of the original evaluation in the MAA Monthly [8].

Example 4. *Recall that*

$$\zeta(\overline{u},v) := \sum_{n=2}^{\infty}\frac{(-1)^n}{n^u}\sum_{k=1}^{n-1}\frac{1}{k^v},\ \textit{while}\ \zeta(u,\overline{v}) := \sum_{n=2}^{\infty}\frac{1}{n^u}\sum_{k=1}^{n-1}\frac{(-1)^k}{k^v},\ \textit{and}$$

$$\zeta(\overline{u},\overline{v}) := \sum_{n=2}^{\infty}\frac{(-1)^n}{n^u}\sum_{k=1}^{n-1}\frac{(-1)^k}{k^v}.$$

The same approach as in Example 1 applied to (5) and (16) produces

$$\frac{1}{\pi}\int_0^\infty \frac{\alpha(3/2+i\tau)\,\overline{\alpha(3/2+i\tau)}}{1/4+\tau^2}\,d\tau = 2\,\zeta(\overline{2},\overline{1}) + \zeta(3) = 3\,\zeta(2)\,\log 2 - \frac{9}{4}\zeta(3),$$

and

$$\frac{1}{2\pi}\int_\infty^\infty \frac{\alpha(3/2+i\tau)\,\overline{\zeta(3/2+i\tau)}}{1/4+\tau^2}\,d\tau = \zeta(\overline{2},1)+\zeta(2,\overline{1})+\alpha(3) = \frac{9}{8}\,\zeta(2)\,\log 2 - \frac{3}{4}\,\zeta(3),$$

as companions to

$$\frac{1}{\pi}\int_0^\infty \frac{\zeta(3/2+i\tau)\,\overline{\zeta(3/2+i\tau)}}{1/4+\tau^2}\,d\tau = 3\,\zeta(3),$$

since, by techniques discussed in [4,5],

$$\zeta(\overline{2},1) = \frac{1}{8}\,\zeta(3), \quad \zeta(2,\overline{1}) = \zeta(3) - \frac{3}{2}\,\zeta(2)\log 2, \quad \zeta(\overline{2},\overline{1}) = \frac{3}{2}\,\zeta(2)\log 2 - \frac{13}{8}\,\zeta(3).$$

4. Final Remarks. Many other similar results obtain. For example:

$$\begin{aligned}\int_{-\infty}^\infty e^{-2\sigma\omega}\{\max(e^\omega - a, 0)\}^2\,d\omega &= \sum_{n=0}^\infty \int_{n+a}^{n+1+a} t^{-2\sigma-1}|\{t-a\}|^2\,dt \\ &= \sum_{n=0}^\infty \int_0^1 \theta^2\,(\theta+n+a)^{-2\sigma-1}\,d\theta \\ &= \int_0^1 \theta^2\,\zeta(1+2\sigma,\theta+a)\,d\theta.\end{aligned}$$

While (4) and (16) give an effective way of evaluating the integral, directly evaluating the integral numerically to high precision presents a greater challenge. This is largely because of the severe oscillations of the integrand. The issue appears to lie in estimating the integrand well and so is intrinsically non-trivial.

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