Dynamics of generalizations of the AGM continued fraction of Ramanujan. Part I: divergence.

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Continued Fractions

For the sequence $a := (a_n)_{n=1}^{\infty}$, denote the continued fraction $\mathcal{S}_1(a)$ by

$$\mathcal{S}_1(a) = \cfrac{1^2 a_1^2}{1 + \cfrac{2^2 a_2^2}{1 + \cfrac{3^2 a_3^2}{1 + \ddots}}}$$

We study the convergence properties of this continued fraction for deterministic and random sequences (a_n). For the deterministic case we derive our most general results from an examination of periodic sequences, that is, sequences satisfying $a_j = a_{j+c}$ for all j and some finite c. Many special cases of the above continued fraction for particular choices of a have been determined in [3, 4]. In particular the cases (i) $a_n = const \in \mathbb{C}$, (ii) $a_n = -a_{n+1} \in \mathbb{C}$, (iii) $|a_{2n}| = 1,\ a_{2n+1} = i$, and (iv) $a_{2n} = a_{2m},\ a_{2n+1} = a_{2m+1}$ with $|a_n| = |a_m|\ \forall\ m, n \in \mathbb{N}$. In the present work we are interested in the convergence of $\mathcal{S}_1$ for arbitrary sequences of parameters.

Difference Equations

To evaluate $\mathcal{S}_1$, we study the recurrence for the classical convergents p_n/q_n to the fraction $\mathcal{S}_1$,

$$p_n = p_{n-1} + n^2 a_n^2 p_{n-2} \quad \text{and} \quad q_n = q_{n-1} + n^2 a_n^2 q_{n-2}$$

Difference Equations

It is helpful to consider the renormalized sequences (t_n) and (v_n) where

$$t_n := \frac{q_{n-1}}{n!} \quad \text{and} \quad v_n := \frac{q_n}{\Gamma(n+3/2)a_n^{(n+1)}}.$$

The corresponding recurrence relations are

$$t_n = \frac{1}{n}t_{n-1} + \frac{n-1}{n}a_{n-1}^2 t_{n-2},$$

and

$$v_n = \frac{2}{a_n(2n+1)}\left(\frac{a_{n-1}}{a_n}\right)^n v_{n-1} + \frac{4n^2}{(2n-1)(2n+1)}\left(\frac{a_{n-2}}{a_n}\right)^{(n-1)} v_{n-2}.$$

Difference Equations

For $|a_n| = |a_m| = b \neq 0$ for all $n, m \in \mathbb{N}$, the continued fraction $\mathcal{S}_1$ diverges – that is, the convergents separate – if

$$|t_n| \leq O\left(\frac{b^n}{\sqrt{n}}\right) \quad \text{or} \quad (v_n) \text{ is bounded,}$$

each of these being equivalent.

Convergence: real parameters

Theorem 1. [arbitrary real parameters] *The generalized Ramanujan continued fraction $\mathcal{S}_1$ converges whenever all parameters a_n are real and satisfy $0 < m \leq |a_n| \leq M < \infty$.*

Issue: What about complex a_n?

Numerical evidence

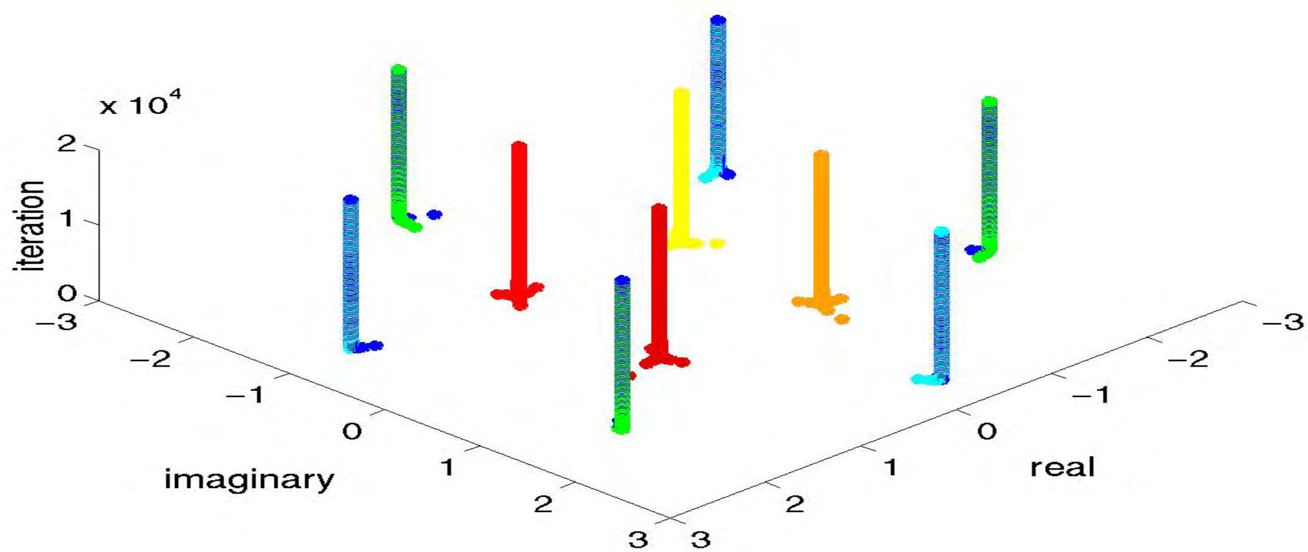


Figure 1: *Dynamics for cycles of length* $c = 2$. *Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* t_n *with* $(a_1, a_2) = (\exp(i\pi/4), \exp(i\pi/6))$. *Odd iterates are light, even iterates are dark.*

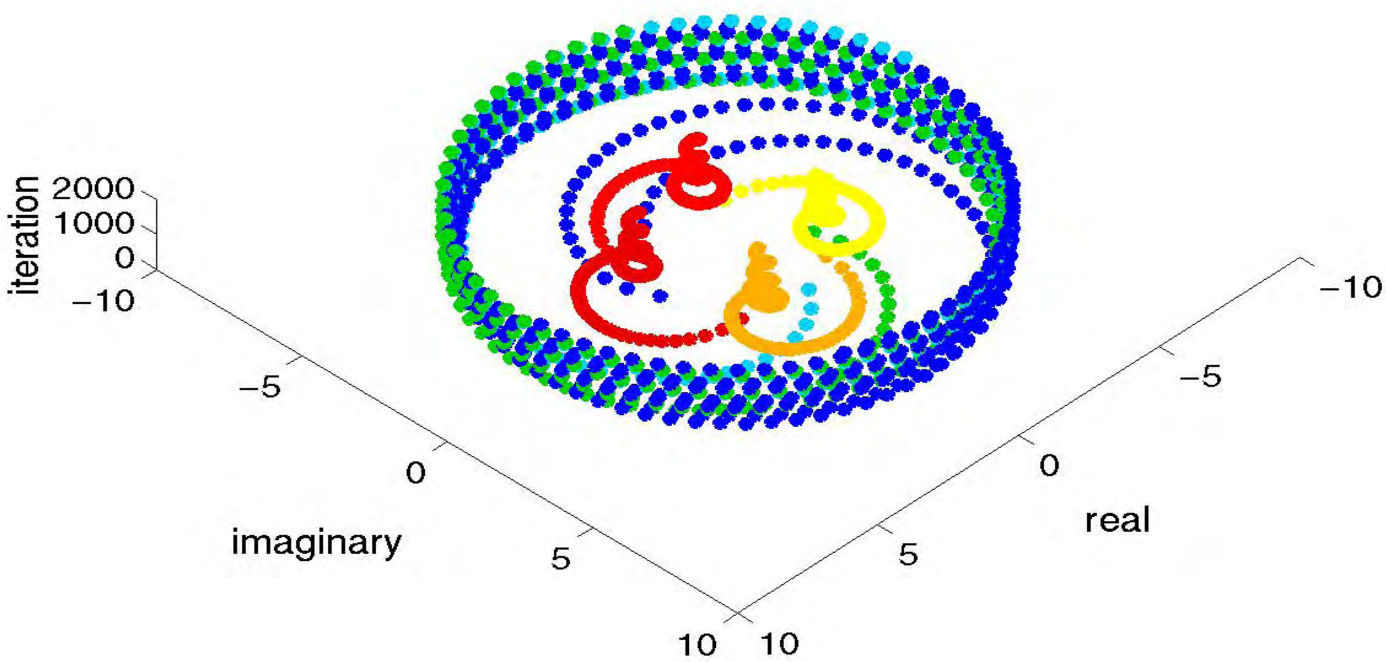


Figure 2: *Dynamics for cycles of length* $c = 4$*. Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* t_n *with cycle length* 4*,* $a_1 = a_3 = \exp(i\pi/4)$*,* $a_2 = \exp(i\pi/6)$*,* $a_4 = \exp(i(\pi/6 + 1/2))$*. Odd iterates are light, even iterates are dark.*

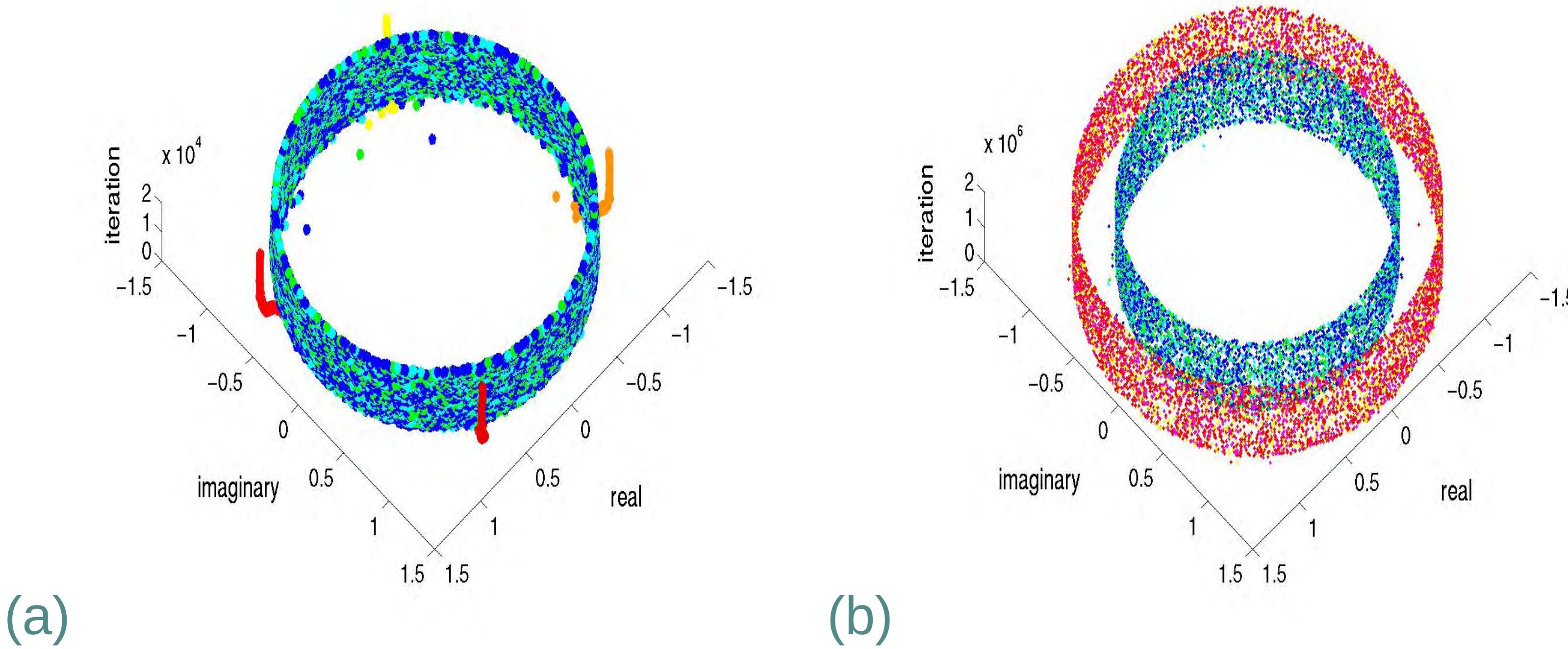


Figure 3: *Dynamics for random cycles. Shown are the iterates* $\widetilde{t}_n := \sqrt{n} t_n$ *for* t_n *with (a) cycle length* ∞ *with only one random strand* $\bmod 2$, $a_{2n+1} = \exp(i\pi/4)$, $a_{2n} = \exp(i\theta_n)$, $\theta_n \sim U[0, 2\pi]$, *and (b) cycle length* ∞ *(i.e.* $a_n = \exp(i\theta_n)$, $\theta_n \sim U[0, 2\pi]$ *for all* n*). Odd iterates are light, even iterates are dark.*

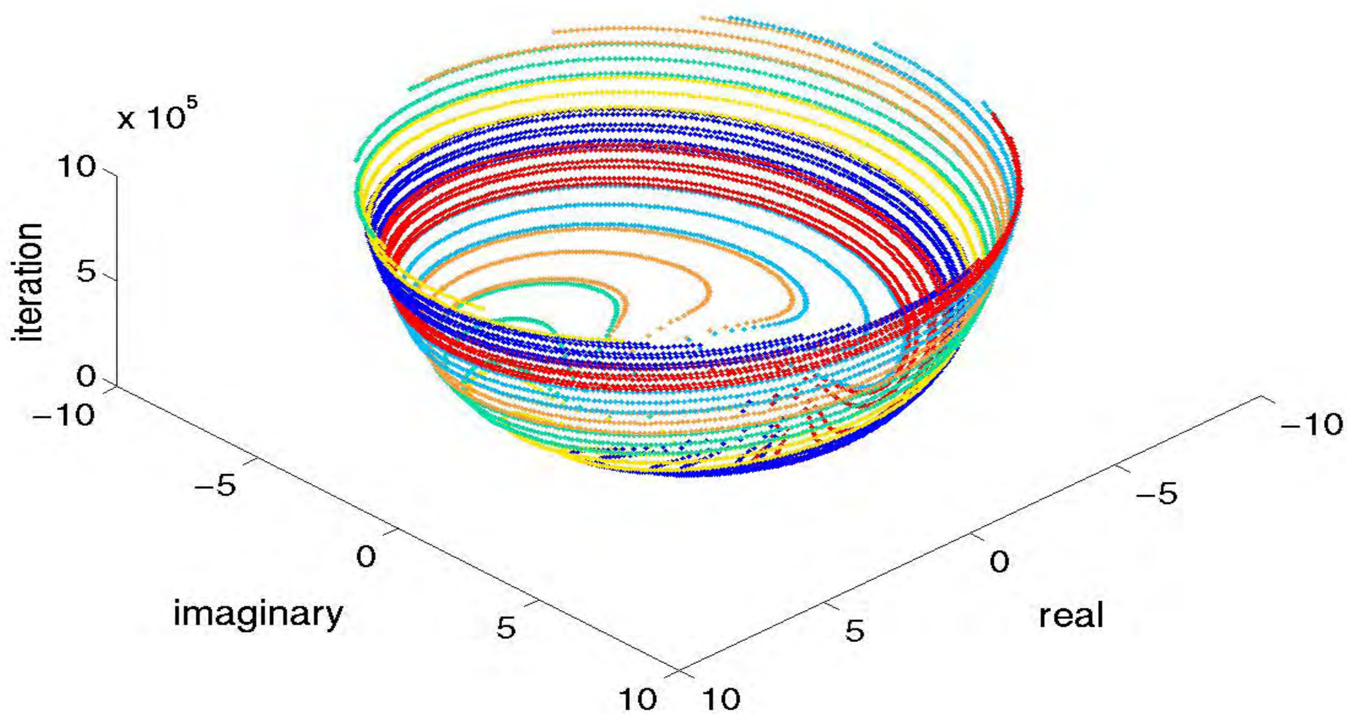
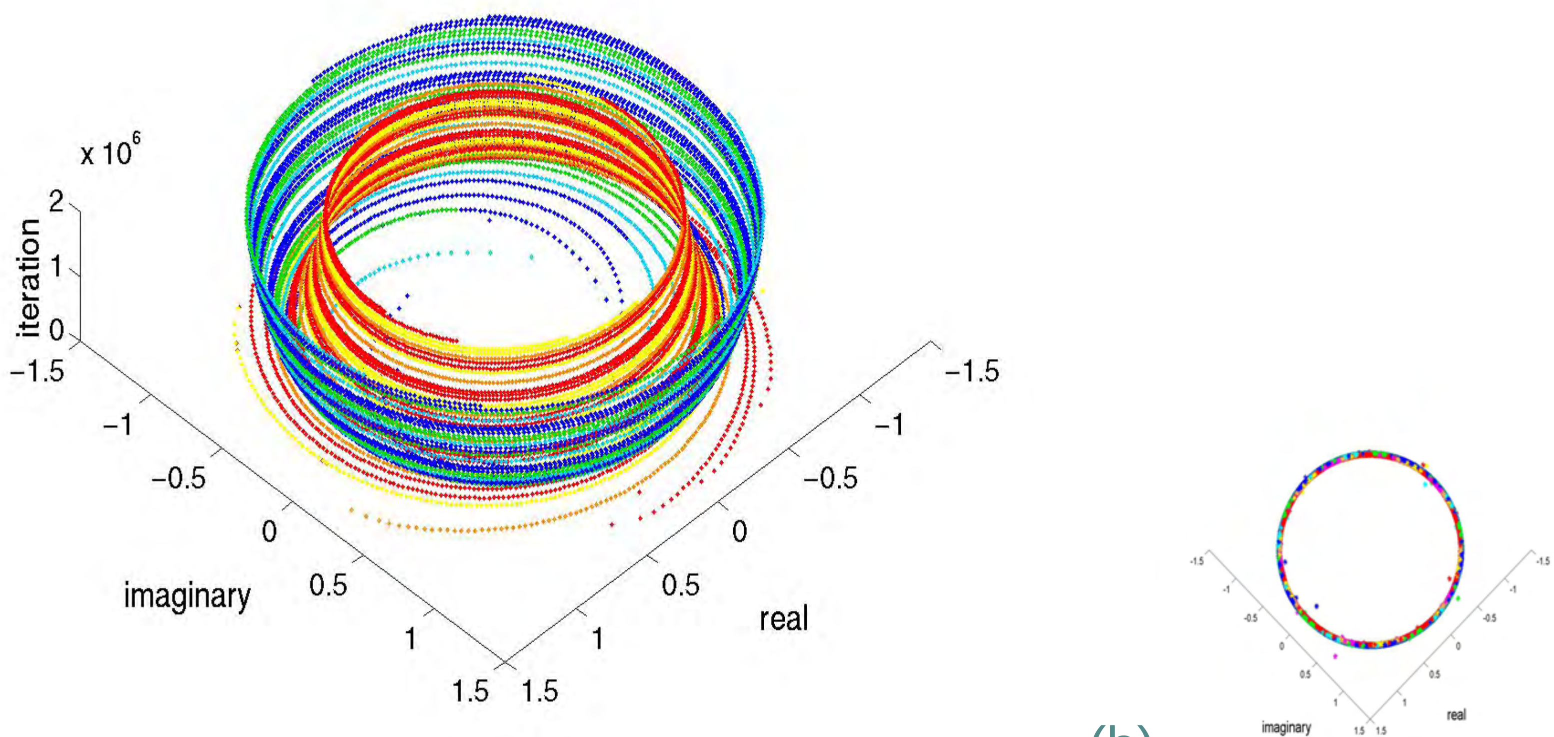


Figure 4: *Dynamics for cycles of length* 3*. Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* t_n *given by with* $(a_1, a_2, a_3) = (\exp(i\pi/4), \exp(i\pi/4), \exp(i\pi/4+1/\sqrt{2}))$*. Odd iterates are light, even iterates are dark.*



(a) (b)

Figure 5: *Dynamics for cycles of length* 3*. Shown are the iterates (a)* $\widetilde{t}_n := \sqrt{n}t_n$ *for* t_n *and (b)* v_n*. In both of these examples the parameter values are* $(a_1, a_2, a_3) = (\exp(i\pi/4), -\exp(i\pi/4), \exp(i\pi/4+1/\sqrt{2}))$*. Odd iterates are light, even iterates are dark.*

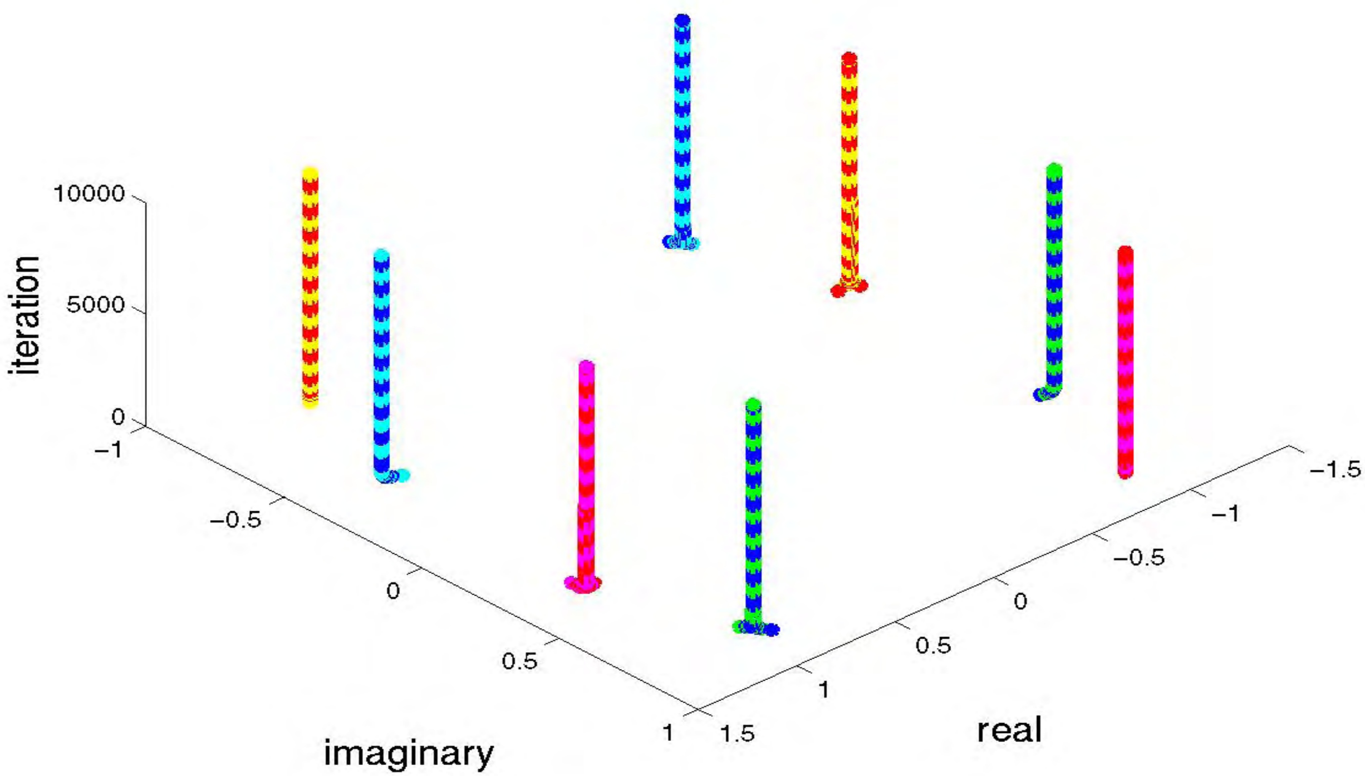


Figure 6: *Dynamics for cycle of length $c = 3$. Shown are the iterates $\widetilde{t}_n := \sqrt{n} t_n$ for t_n with $(a_1, a_2, a_3) = (\exp(i\pi/2), \exp(i\pi/6), \exp(-i\pi/6))$. Even iterates are light, odd iterates are dark.*

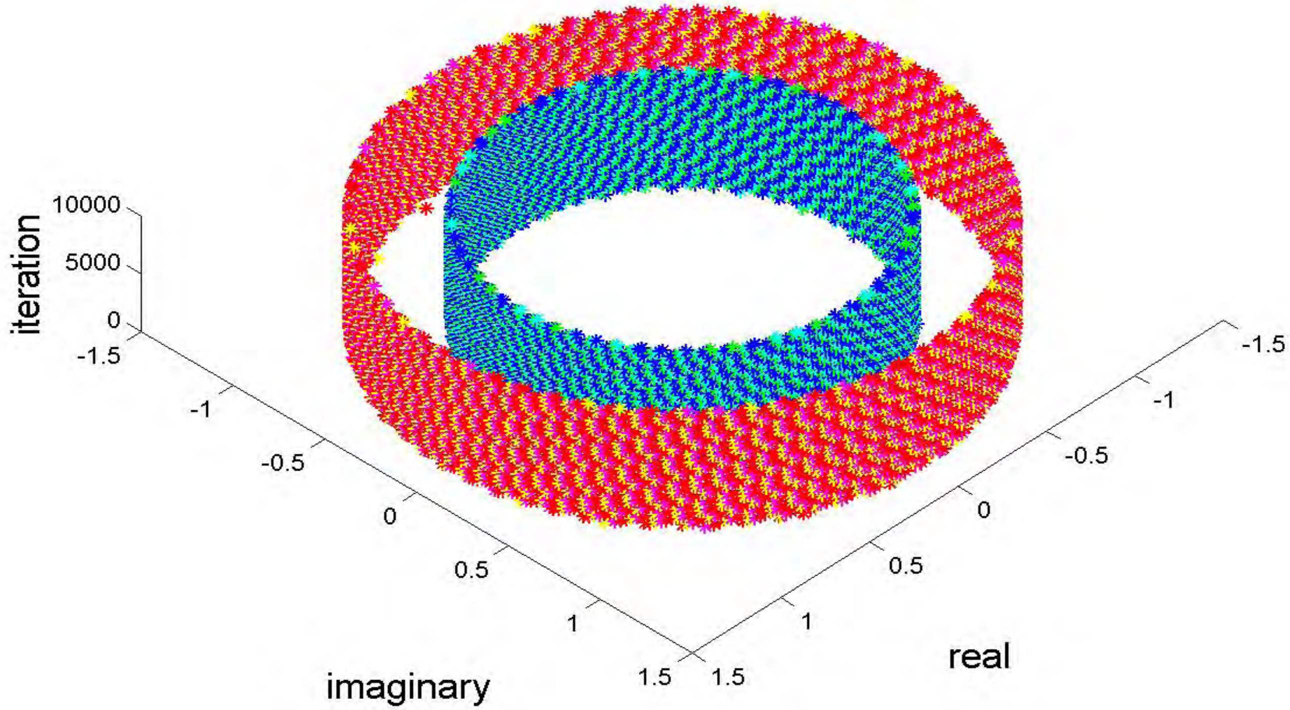


Figure 7: *Dynamics for cycle of length* $c = 3$. *Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* t_n *with* $(a_1, a_2, a_3) = (\exp(i(\pi/3 + 0.05)), \exp(-i(\pi/3 + 0.05)), \exp(0.05i))$. *Even iterates are light, odd iterates are dark.*

Convergence/divergence: general (random) parameters

Theorem 2. [summary] *Let the nonzero (random) complex sequence of parameters* $a := (a_n)$ *satisfy (in probability)*

$$0 \neq \prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n)^2 a_{2n}^2}\right) < \infty \quad \textit{and} \quad 0 \neq \lim_{n\to\infty} \frac{a_2}{a_{2n}^{2n-1} a_{2n-1}^{2n-2}} \prod_{j=1}^{2n-2} a_j^2 < \infty.$$

The iterates v_n *of the corresponding (stochastic) difference equation are bounded (with probability* 1*) and the (stochastic) Ramanujan continued fraction* $\mathcal{S}_1(a)$ *diverges (almost surely) with the even/odd parts of* $\mathcal{S}_1(a)$ *converging (in probability) to separate limits in the following cases:*

Convergence/divergence: general (random) parameters

(i) Even periodic parameters: *If* $a_n = a_{n+c}$ *for all* n *and fixed* c *even, and* $|\gamma| = 1$ *with* $\gamma \neq 1$ *where*

$$\gamma := \left(\prod_{n=1}^{c/2} \frac{a_{2n-1}^2}{a_{2n}^2} \right).$$

(ii) General deterministic parameters:

$$\sup_k \left| \sum_{j \geq n}^{k} \frac{1}{a_2} \prod_{i=1}^{j} \frac{a_{2i-1}^2}{a_{2i}^2} \right| < \infty \quad \textit{and} \quad \sup_k \left| \sum_{j \geq n}^{k} \frac{a_2}{a_{2j}^2} \prod_{i=1}^{j} \frac{a_{2i}^2}{a_{2i-1}^2} \right| < \infty.$$

(iii) Random parameters:

$$\sum_n^{\infty} \frac{1}{n^2} \textit{var} \left(\frac{1}{a_2} \prod_{j=1}^{n} \frac{a_{2j-1}^2}{a_{2j}^2} \right) < \infty \quad \textit{and} \quad \sum_n^{\infty} \frac{1}{n^2} \textit{var} \left(\frac{a_2}{a_{2n}^2} \prod_{j=1}^{n} \frac{a_{2j}^2}{a_{2j-1}^2} \right) < \infty.$$

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