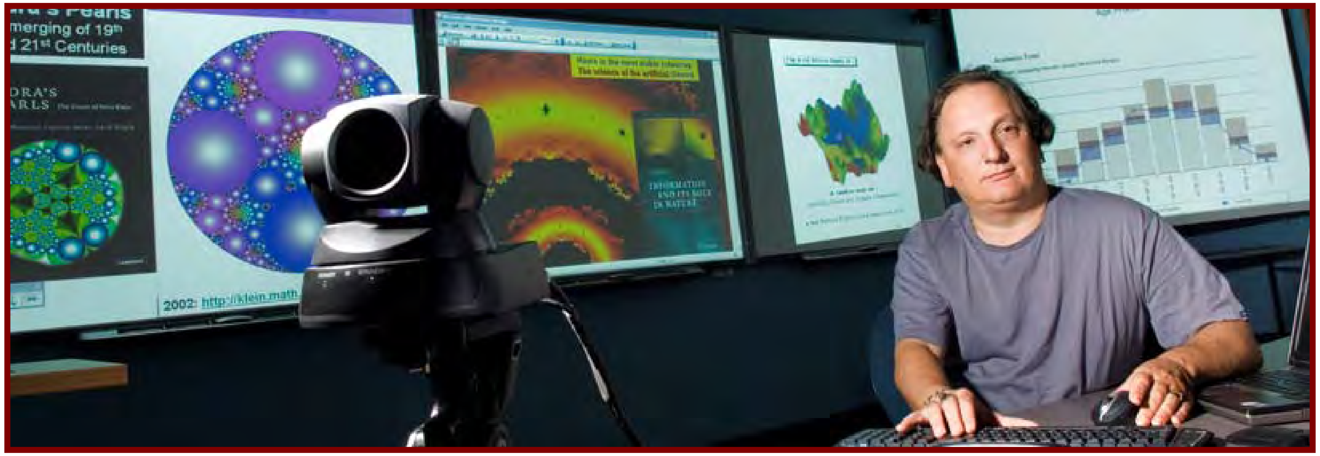

Newcastle
AMSI-AG Room

Convex functions:
Characterizations, Constructions and Counterexamples

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A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs; and the best mathematician can notice analogies between theories.

(Stefan Banach, 1892-1945)

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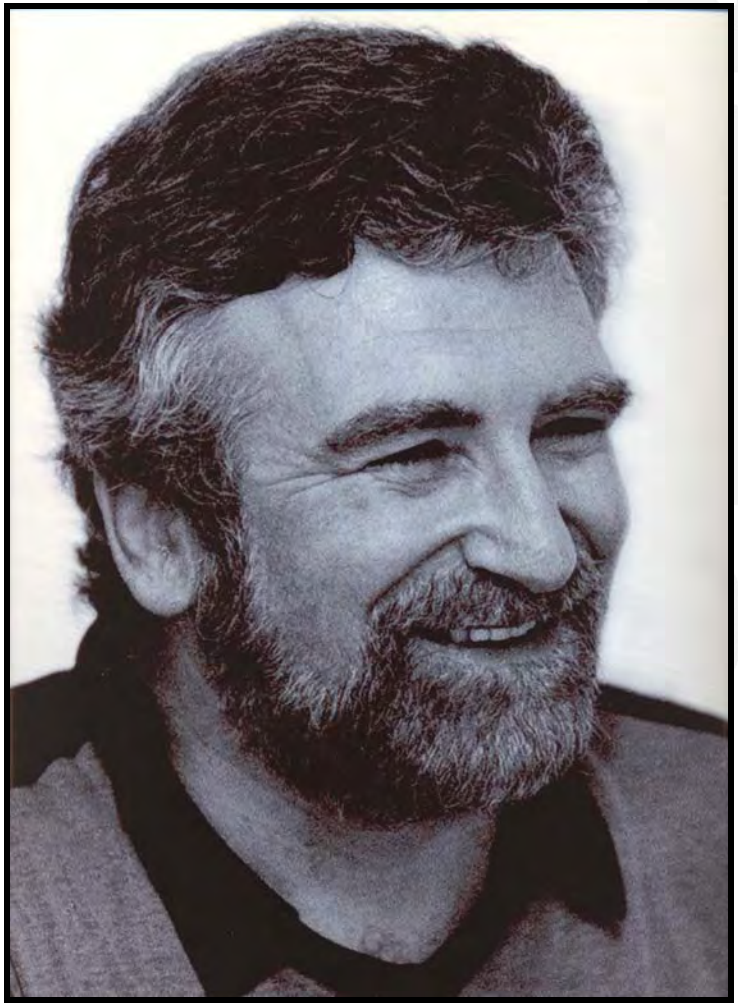
01/09/2008

Abstract of CF:CCC Talk

In honour of my friend Boris Mordhukovich



We met in 1990. He said

"How old are you?"

I said *"39 and you?"*

He replied "48."

I left thinking he was 48 and he thinking I was 51.

Some years later Terry Rockafellar corrected our cultural misconnect.

What was it?

Convex Functions: Characterizations, Constructions and Counter examples

(CUP in press)

Convex functions, along with smooth functions, provide the wellspring for much of variational analysis

In this talk I shall look at four open problems in variational analysis, at the convex structure underlying them, and at the convex tools available to make progress with them

In each case, I think better understanding is fundamental to advancing nonsmooth analysis

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Convex Functions

- H is (separable) Hilbert space in some renorm $\|\cdot\|$
- C is a norm-closed subset and

$$d_C(x) := \inf_{x \in C} \| x - c \|$$

$$P_C(x) := \arg\min d_C(x)$$

- In the Hilbert case $P_C(x)$ is at most singleton
- In a nonrotund renorm it may be multivalued
- If C is convex it is non-empty

Most of these questions that follow are no easier in arbitrary renorms of Hilbert space than in reflexive Banach space

The Chebyshev problem (Klee 1961)

If every point in H has a unique nearest point in C is C convex?

Existence of nearest points (proximal boundary?)

Do some (many) points in H have a nearest point in C in every renorm of H ?

Second-order expansions in separable Hilbert space

If f is convex and continuous on H does f have a second order Taylor expansion at some (many) points?

Universal barrier functions in infinite dimensions

Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

Convex Functions

The Chebyshev problem (Klee 1961) A set is Chebyshev if every point in H has a unique nearest point in C

Theorem If C is weakly closed and Chebyshev then C is convex. So in Euclidean space Chebyshev iff convex.

Four Euclidean variational proofs (BL 2005, Opt Letters 07, BV 2008)

1. **Brouwer's theorem** (Cheb. implies **sun** implies convex)
2. **Ekeland's theorem** (Cheb. implies **approx. convex** implies convex)
3. **Fenchel duality** (Cheb. iff d_C^2 is Frechet) use f^* smooth implies f convex for

$$\left(\frac{\iota_C + \|\cdot\|^2}{2}\right)^* = \frac{\|\cdot\|^2 + d_C^2}{2}$$

4. Inverse geometry also shows if there is a counter-example it can be a **Klee cavern** (Asplund) the closure of the complement of a convex body. **WEIRD**

- <u>**Counterexamples**</u> exist in incomplete inner product spaces. #2 seems most likely to work in Hilbert space.
- **Euclidean case** is due to Motzkin-Bunt

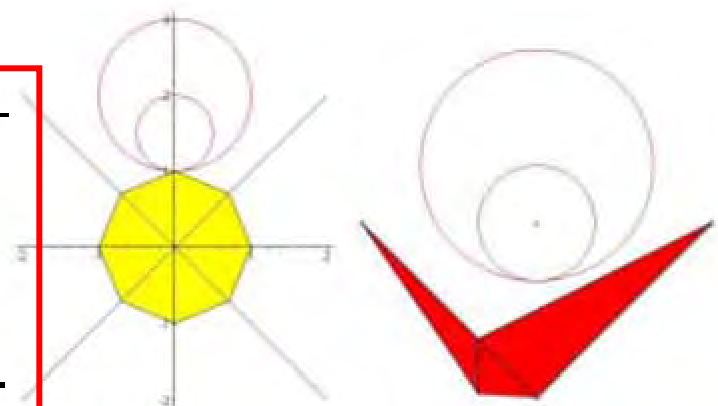


FIGURE 1. Suns and approximate convexity.

Existence of nearest points

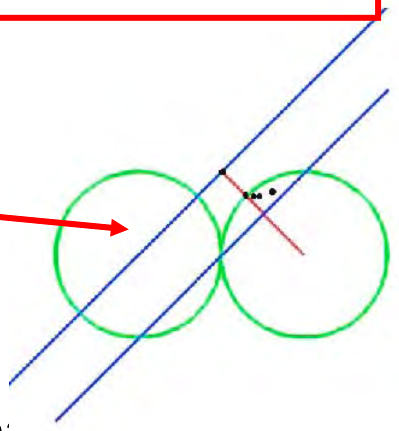
Do some (many) points in H have a nearest point in C in every renorm of H ?

Theorem (Lau-Konjagin, 76-86) A norm on a reflexive space is Kadec-Klee iff for every norm-closed C in X best approximations exist generically (densely) in X\ C

Nicest proof is via dense existence of Frechet subderivatives

$$\varphi \in \partial_F d_C(x)$$

The KK property forces approximate minimizers to line up.



- There are non KK norms with proximal points dense in bdry C
- If C is closed and bounded then there are some points with nea…
- So a counterexample has to be a weird unbounded set in a rotten renorm (BF89, BZ 2005)

A norm is **Kadec-Klee** norm if weak and norm topologies agree on the unit sphere.

Hence all LUR norms are Kadec-Klee.

Second-order derivatives in separable Hilbert space

If f is continuous and convex on H does f have a (weak) second-order Taylor expansion at some (many) points?

Theorem (Alexandrov) In Euclidean space the points at which a continuous convex function admits a second-order Taylor expansion are full measure

- In Banach space, this is known to fail pretty completely unless one restricts the class of functions, say to nice integral functionals
- It is possible in separable Hilbert space (BV 2009) that every such f has at least one point with a second-order Gateaux expansion?
- The goal is to build good jets and save as much as possible of extensions of lovely Euclidean results like

$$\partial\left[\frac{1}{2}\Delta_t^2 f(x)\right] = \Delta_t[\partial f](x).$$

Universal barrier functions in infinite dimensions

– Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

Theorem (Nesterov-Nemirovskii) For any open convex set A in n-space, the function

$$F(x) := \lambda_N((A-x)^o)$$

is an essentially smooth, log-convex barrier function for A.

– This relies heavily on the existence of **Haar** measure (Lebesgue).
– Amazingly for A the semidefinite matrix cone we recover – log det, etc

In Hilbert space the only really nice examples I know are similar to:

$$\phi(T) := \mathrm{trace}(T) - \log(\det(I+T))$$

is a strictly convex Frechet differentiable barrier function for the Hilbert-Schmidt operators with I+T > 0.

The Chebyshev problem (Klee 1961)

If every point in H has a unique nearest point in C is C convex?

I HAVE A SUGGESTION FOR THESE TWO: DISTORTION

Existence of nearest points (proximal boundary?)

Do some (many) points in H have a nearest point in C in every renorm of H

Second-order expansions in separable Hilbert space

If f is convex continuous on H does f have a second order Taylor expansion at some (many) points?

I THINK PROGRESS FOR THESE TWO WILL BE INCREMENTAL

Universal barrier functions in infinite dimensions

Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

A Banach space X is **distortable** if there is a renorm and $\lambda>1$ such that, for all infinite-dimensional subspaces $Y \subseteq X$,

$$\sup\{|y|/|x| \; x, y \in Y, \|x\| = \|y\| = 1\} > \lambda.$$

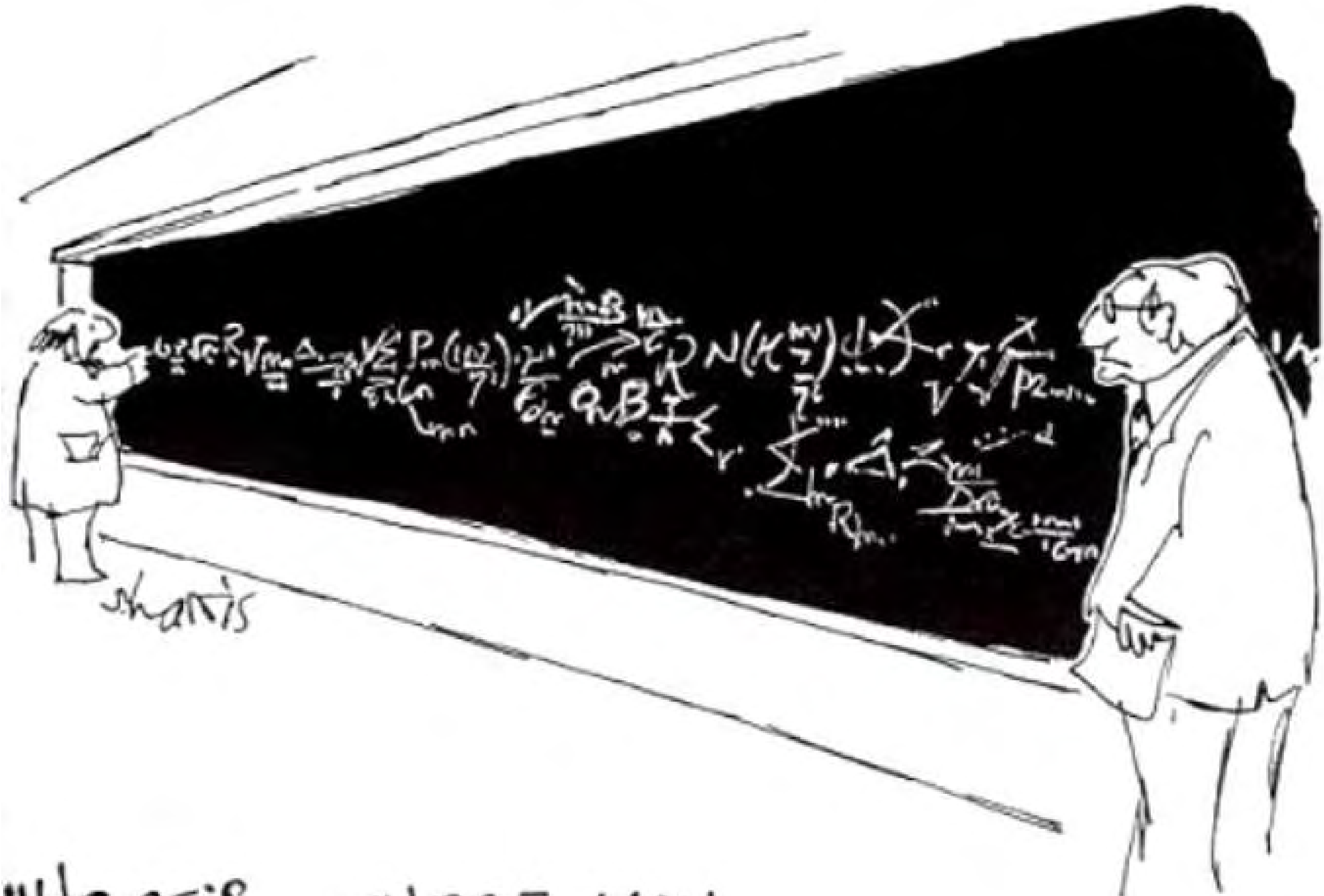
X is **arbitrarily distortable** if this can be done for all $\lambda>1$.

Theorem (Odell and Schlumprecht 93,94) Separable infinite-dimensional Hilbert space is arbitrarily distortable

Distortability of $l_2(N)$ is equivalent to existence of two separated sets in the sphere both intersecting every infinite-dimensional closed subspace of $l_2(N)$. Indeed, there is a sequence of (**asymptotically orthogonal**) subsets $(C_i)_{i=1}^{\infty}$ of the unit sphere such that (a) each set C_i intersects each infinite-dimensional closed subspace of and (b) as $i,j \to \infty$

$$\sup\{|\langle x, y\rangle| \; x \in C_i, y \in C_j\} \to 0$$

These are such surprising sequences of sets that they should shed insight on the two proximality questions



"<u>HERE'S</u> WHERE YOU MADE YOUR MISTAKE."

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"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

- J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.