Convexity on groups and semigroups

Jonathan M. Borwein   Ohad Giladi

University of Newcastle

August 23, 2015

Outline

- Convex sets/functions
- Examples
- Convex Analysis
- Future directions

General theme

Many known results hold assuming only an additive structure

Definition (Convex set in vector spaces)

X a vector space. $A \subseteq X$ is convex if $x_1, \ldots, x_n \in A$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n \alpha_i x_i \in A$.

Definition (Convex set in vector spaces)

X a vector space. $A \subseteq X$ is convex if $x_1, \ldots, x_n \in A$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1 \Longrightarrow \sum_{i=1}^n \alpha_i x_i \in A$.

If $\alpha_i \in \mathbb{Q}$, write $\alpha_i = \frac{m_i}{m}$. Then $\sum_{i=1}^n \alpha_i = 1 \Longrightarrow \sum_{i=1}^n m_i = m$.

Definition (Convex set in vector spaces)

X a vector space. $A \subseteq X$ is convex if $x_1, \ldots, x_n \in A$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n \alpha_i x_i \in A$.

If $\alpha_i \in \mathbb{Q}$, write $\alpha_i = \frac{m_i}{m}$. Then $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n m_i = m$.

Monoid $=$ additive semigroup with unit.

Definition (Convex set in vector spaces)

X a vector space. $A \subseteq X$ is convex if $x_1, \ldots, x_n \in A$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n \alpha_i x_i \in A$.

If $\alpha_i \in \mathbb{Q}$, write $\alpha_i = \frac{m_i}{m}$. Then $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n m_i = m$.

Monoid = additive semigroup with unit.

Definition (Convex set in monoids/groups)

X a monoid. $A \subseteq X$ is said to be convex if $x_1, \ldots, x_n \in A$, $m_1, \ldots, m_n, m \in \mathbb{N}$, $mx = \sum_{i=1}^n m_i x_i$, $m = \sum_{i=1}^n m_i \implies x \in A$.

Definition (Convex set in vector spaces)

X a vector space. $A \subseteq X$ is convex if $x_1, \ldots, x_n \in A$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n \alpha_i x_i \in A$.

If $\alpha_i \in \mathbb{Q}$, write $\alpha_i = \frac{m_i}{m}$. Then $\sum_{i=1}^n \alpha_i = 1 \implies \sum_{i=1}^n m_i = m$.

Monoid = additive semigroup with unit.

Definition (Convex set in monoids/groups)

X a monoid. $A \subseteq X$ is said to be convex if $x_1, \ldots, x_n \in A$, $m_1, \ldots, m_n, m \in \mathbb{N}$, $mx = \sum_{i=1}^n m_i x_i$, $m = \sum_{i=1}^n m_i \implies x \in A$.

Definition (Convex hull)

For $A \subseteq X$, $\mathrm{conv}(A)$ is the smallest convex set that contains A.

Convex function

Definition (Convex function on vector spaces)

X a vector space. $f : X \to \mathbb{R}$ is convex if $f(x) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$, whenever $x = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$. f is concave if $-f$ is convex

Convex function

Definition (Convex function on vector spaces)

X a vector space. $f : X \to \mathbb{R}$ is convex if $f(x) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$, whenever $x = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$. f is concave if $-f$ is convex

Definition (Convex function on monoids/groups)

X a monoid. $f : X \to \mathbb{R}$ is convex if $mf(x) \leq \sum_{i=1}^{n} m_i f(x_i)$, whenever $mx = \sum_{i=1}^{n} m_i x_i$, $m = \sum_{i=1}^{n} m_i$. f is concave if $-f$ is convex

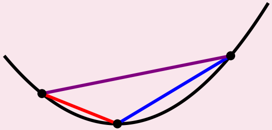
This can be done in a more general setting (X is a module, $\pm\infty$...)

Example of some basic properties

Example: three slopes lemma

X a monoid, $f : X \to \mathbb{R}$ convex, $m, m_1, m_2 \in \mathbb{N}$, $x, x_1, x_2 \in X$ such that $mx = m_1x_1 + m_2x_2$. Then

$$\frac{f(x) - f(x_1)}{m_2} \leq \frac{f(x_2) - f(x_1)}{m_1 + m_2} \leq \frac{f(x_2) - f(x)}{m_1}$$



Divisible and semidivisible monoids/groups

In some cases, want to solve the equation $mx = \sum_{i=1}^{n} m_i x_i$, at least for some $m \in \mathbb{N}$.

Divisible and semidivisible monoids/groups

In some cases, want to solve the equation $mx = \sum_{i=1}^{n} m_i x_i$, at least for some $m \in \mathbb{N}$.

Definition (Divisible, semidivisible monoids/groups)

A monoid/group X is p-semidivisible if $pX = X$. X is semidivisible if it is p-semidivisible for some p. X is divisible if it is p-semidivisible for every p.

Divisible and semidivisible monoids/groups

In some cases, want to solve the equation $mx = \sum_{i=1}^{n} m_i x_i$, at least for some $m \in \mathbb{N}$.

Definition (Divisible, semidivisible monoids/groups)

A monoid/group X is p-semidivisible if $pX = X$. X is semidivisible if it is p-semidivisible for some p. X is divisible if it is p-semidivisible for every p.

p-semidivisible: same as saying that for every $x \in X$, there exists $y \in X$ such that $x = py$.

Convex sets in certain groups

Finite groups

For every $x \in X$, there is m such that $mx = 0 \Longrightarrow \mathrm{conv}(\{0\}) = X$.
The only convex sets are X and $\emptyset$.

Convex sets in certain groups

Finite groups

For every $x \in X$, there is m such that $mx = 0 \implies \mathrm{conv}(\{0\}) = X$. The only convex sets are X and $\emptyset$.

Circle group

$X = \mathbb{R}/\mathbb{Z}$. Then $\mathrm{conv}(\{x\}) = \{x + y \mid y \in \mathbb{Q}\} \implies$ no convex singletons in X.

Convex sets in certain groups

Finite groups

For every $x \in X$, there is m such that $mx = 0 \Longrightarrow \mathrm{conv}(\{0\}) = X$. The only convex sets are X and $\emptyset$.

Circle group

$X = \mathbb{R}/\mathbb{Z}$. Then $\mathrm{conv}(\{x\}) = \{x + y \mid y \in \mathbb{Q}\} \Longrightarrow$ no convex singletons in X.

Integer lattice $\mathbb{Z}^d$

For $A \subseteq \mathbb{Z}^d$, $\mathrm{conv}_{\mathbb{Z}^d}(A) = \mathrm{conv}_{\mathbb{R}^d}(A) \cap \mathbb{Z}^d$.

Convex sets in certain groups

Arctan semigroup

$X = [0, \infty)$ with the addition $a \oplus b = \frac{a+b}{1+ab}$. If $a, b \neq 0$ then $\frac{1}{a} \oplus \frac{1}{b} = a \oplus b$. Thus, if $a \neq 1$, then $\frac{1}{a} \in \mathrm{conv}(\{a\})$, and so $\{0\}$, $\{1\}$ are the only convex singletons. X is 3-semidivisible but not 2-semidivisible.

Convex sets in certain groups

Arctan semigroup

$X = [0, \infty)$ with the addition $a \oplus b = \frac{a+b}{1+ab}$. If $a, b \neq 0$ then $\frac{1}{a} \oplus \frac{1}{b} = a \oplus b$. Thus, if $a \neq 1$, then $\frac{1}{a} \in \mathrm{conv}(\{a\})$, and so $\{0\}$, $\{1\}$ are the only convex singletons. X is 3-semidivisible but not 2-semidivisible.

Hyperbolic group

Let X be the matrices of the form $\pm \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}$, $\theta \in \mathbb{R}$, with the matrix multiplication. Then $2nX \neq X$ and $(2n+1)X = X$. If $f : \mathbb{R} \to \mathbb{R}$ is convex then

$$F\left(\pm \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}\right) = f(\theta) \text{ is convex.}$$

Interpolation of convex functions

$f : X \to \mathbb{R}$ is subadditive if $f(x+y) \leq f(x) + f(y)$.

Theorem (Kaufman)

X a monoid, $f, -g : X \to \mathbb{R}$ subadditive, and $g \leq f$. Then there exists $a : X \to \mathbb{R}$ additive such that $g \leq a \leq f$.

This is a monoid version of result by Mazur-Orlicz

Interpolation of convex functions

$f : X \to \mathbb{R}$ is subadditive if $f(x+y) \leq f(x) + f(y)$.

Theorem (Kaufman)

X a monoid, $f, -g : X \to \mathbb{R}$ subadditive, and $g \leq f$. Then there exists $a : X \to \mathbb{R}$ additive such that $g \leq a \leq f$.

This is a monoid version of result by Mazur-Orlicz

$f : X \to \mathbb{R}$ is affine if it is both convex and concave.

Interpolation of convex functions

$f : X \to \mathbb{R}$ is subadditive if $f(x+y) \le f(x) + f(y)$.

Theorem (Kaufman)

X a monoid, $f, -g : X \to \mathbb{R}$ subadditive, and $g \le f$. Then there exists $a : X \to \mathbb{R}$ additive such that $g \le a \le f$.

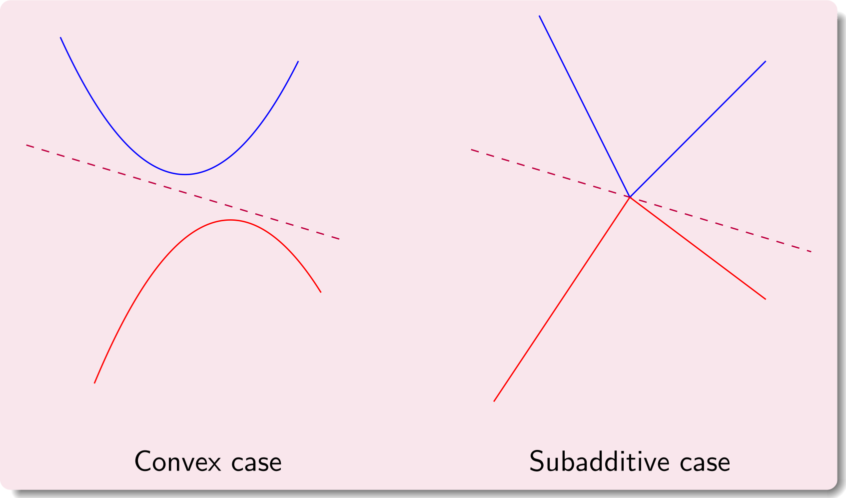
This is a monoid version of result by Mazur-Orlicz

$f : X \to \mathbb{R}$ is affine if it is both convex and concave.

Theorem

X is a semidivisible monoid, $f, -g : X \to \mathbb{R}$ convex, and $g \le f$. Then there exists $a : X \to \mathbb{R}$ affine such that $g \le a \le f$.

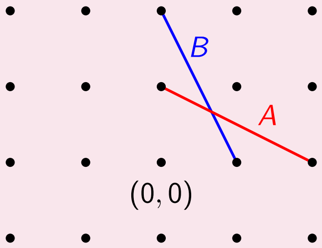
Picture: interpolation of subadditive/convex functions



Example: nondivisible case

Failure in the nondivisible case

$X = \mathbb{Z}^2$, $f(x) = 5d_A(x) - 1$ and $g = -5d_B(x) + 1$.



$f, -g$ are convex, $g \leq f$, but there is no affine a s.t. $g \leq a \leq f$.

Directional derivative and subgradient

Definition (Directional derivative)

$$f_x(h) = \inf \left\{ n\big(f(x+g) - f(x)\big) \;\middle|\; ng = h \right\}$$

If f is convex: $n\big(f(x+g) - f(x)\big)$ is decreasing in n.

Directional derivative and subgradient

Definition (Directional derivative)

$$f_x(h) = \inf \left\{ n\big(f(x+g) - f(x)\big) \;\middle|\; ng = h \right\}$$

If f is convex: $n\big(f(x+g) - f(x)\big)$ is decreasing in n.
Recall, if X is a VS: $f_x(h) = \inf \left\{ \frac{1}{t}\big(f(x+th) - f(x) \;\middle|\; t > 0\big) \right\}$.

Directional derivative and subgradient

Definition (Directional derivative)

$$f_x(h) = \inf \left\{ n\big(f(x+g) - f(x)\big) \;\middle|\; ng = h \right\}$$

If f is convex: $n\big(f(x+g) - f(x)\big)$ is decreasing in n.
Recall, if X is a VS: $f_x(h) = \inf \left\{ \frac{1}{t}\big(f(x+th) - f(x) \;\middle|\; t > 0\big) \right\}$.

Definition (Subgradient)

$$\partial f(x) = \left\{ a : X \to \mathbb{R} \;\middle|\; f(x) + a(h) \leq f(x+h), a \text{ additive} \right\}$$

Directional derivative and subgradient

Definition (Directional derivative)

$$f_x(h) = \inf\left\{n\big(f(x+g) - f(x)\big) \;\middle|\; ng = h\right\}$$

If f is convex: $n\big(f(x+g) - f(x)\big)$ is decreasing in n.
Recall, if X is a VS: $f_x(h) = \inf\left\{\frac{1}{t}\big(f(x+th) - f(x)\;\middle|\; t > 0\big)\right\}$.

Definition (Subgradient)

$$\partial f(x) = \left\{a : X \to \mathbb{R} \;\middle|\; f(x) + a(h) \le f(x+h), a \text{ additive}\right\}$$

Theorem (Max formula)

X a semidivisible group and $f : X \to \mathbb{R}$ convex. Then

$$f_x(h) = \max\left\{a(h) \;\middle|\; a \in \partial f(x)\right\}$$

Consequences of the max formula

Definition (Sublinear function)

$f : X \to \mathbb{R}$ is sublinear if $f(nx) = nf(x)$ and f is subadditive.

Theorem (Hahn-Banach for groups)

X a group and $Y \subseteq X$ a subgroup. $f : X \to \mathbb{R}$ is sublinear and $h : Y \to \mathbb{R}$ is additive such that $h \le f$ on Y. Then there exists $\bar{h} : X \to \mathbb{R}$ additive such that $\bar{h} \le f$ and $\bar{h} = h$ on Y.

Did not use semidvisibility since the functions are sublinear.

Convex optimisation on groups

Consider the constrained problem

$$\inf\left\{f(x) \;\middle|\; g_1(x) \le 0, \ldots, g_k(x) \le 0\right\}$$

Theorem (Subgradient of max function)

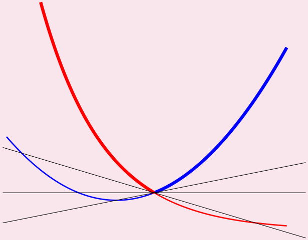
X semidivisible group and $f_1, \ldots, f_k : X \to \mathbb{R}$ convex. Let $g(x) = \max_{1 \le i \le k} f_i(x)$. Then

$$\partial g(x) = \operatorname{conv}\Bigg(\bigcup_{f_i(x)=g(x)} \partial f_i(x) \Bigg)$$

Convex optimisation on groups

Subgradient of max function

$$\partial g(x) = \mathrm{conv}\Bigg(\bigcup_{f_i(x)=g(x)} \partial f_i(x) \Bigg), \quad g = \max_{1 \le i \le k} f_i$$



Future Directions

- Noncommutative groups
- Questions in topological groups (continuity, differentiability...)
- Applications in integer programming

The End