A Characterisation of the Non-strict Opial Condition

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Dedicated to Brailey Sims on the occasion of his retirement

1. Introduction

The Opial conditions come in three versions; uniform Opial condition, Opial's condition and non-strict Opial condition. The first implies the second which, in turn, implies the third. They all have been an important part of the research into fixed points of nonexpansive maps in Banach spaces and also in the study of conditions known to imply the existence of such fixed points. The Opial conditions describe how, in Banach spaces, the weak and strong topologies in Banach space may intertwine.

In 1967 Opial [7] introduced the following condition on a Banach space X.

If (x_n) converges weakly to $x_\infty, x_n \rightharpoonup x_\infty$, then

$$\liminf_{x\to\infty} \|x_n - x_\infty\| < \liminf_{x\to\infty} \|x_n - x\| \text{ for all } x \neq x_\infty.$$

In 1992 Prus [8] defined the uniform Opial condition via a Opial modulus.

For $c > 0$ define the Opial modulus of X to be

$$r(c) = \inf\{\liminf_n \|x_n - x\| - 1 : \|x\| \geq c, x_n \rightharpoonup 0 \text{ and } \liminf_n |x_n\| \geq 1\}.$$

X has the uniform Opial condition if $r(c) > 0$ for $c > 0$.

The non-strict Opial condition, which cannot be attributed to any particular author, states

If (x_n) converges weakly to x_∞, then

$$\liminf_{x\to\infty} \|x_n - x_\infty\| \leq \liminf_{x\to\infty} \|x_n - x\| \text{ for all } x \neq x_\infty.$$

Dalby and Sims [3] characterised the non-strict Opial condition in terms of the duality map. This paper presents another characterisation in terms of weak null types.

Here a weak null type is $\psi_{(x_n)}(x) := \limsup_n \|x - x_n\|$ where $x_n \rightharpoonup 0$. So non-strict Opial means

$$\psi_{(x_n)}(0) \leq \psi_{(x_n)}(x) \text{ for all } x \in X.$$

This characterisation was partially completed by Garcia Falset and Sims [5]. The next section discusses the background to that paper.

2. Property (M) and WORTH

Kalton [6] introduced property(M) in Banach spaces. This property states that whenever $x_n \rightharpoonup 0, \limsup_n \|x - x_n\|$ is a function of $\|x\|$ only. Kalton used this property in his work on characterising those separable Banach spaces whose compact operators form an M-ideal in the algebra of all bounded linear operators.

Borwein and Sims [1] looked at the weak fixed point property in Banach lattices, including c_0, using the lattice property of weak orthogonality. This research led Sims [9] to generalise that Banach lattice property to Banach spaces. This is called WORTH:

$$\text{if } x_n \rightharpoonup 0 \text{ then } \limsup_n \|x - x_n\| = \limsup_n \|x + x_n\| \text{ for all } x \in X$$

Clearly property(M) implies WORTH. Recently it has been shown, [4] and [2], that when the property WORTH is translated to dual Banach spaces and weak* null sequences are used in the definition then X has the weak fixed point property.

Garcia Falset and Sims [5] were able to show that property(M) implies the weak fixed point property. Near the beginning of that paper is the following proposition.

PROPOSITION 1.1: *For the following conditions on the Banach space X we have (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv).*

(i) X has property(M).

(ii) X has WORTH.

(iii) If $x_n \rightharpoonup 0$ then for each $x \in X$ we have $\psi_{(x_n)}(tx)$ an increasing function of t on $[0, \infty)$.

(iv) X satisfies the non-strict Opial property.

The next section shows that (iv) $\implies$ (iii) and thus completing the characterisation.

3. The reverse implication

PROPOSITION: *If a separable Banach space X has the non-strict Opial condition then for every $x_n \rightharpoonup 0$ and every $x \in X$ we have $\psi_{(x_n)}(tx)$ an increasing function of t on* $[0, \infty)$.

PROOF: Assume X is a separable Banach space, $x_n \rightharpoonup 0, x \in X, x \neq 0,$ and $t > 0$. So $tx - x_n \rightharpoonup tx \neq 0$.

The mapping $J : X \to X^*$ is called the duality mapping if for every $x \in X$

$$J(x) := \{x^* \in X^* : x^*(x) = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}.$$

In [3] it was shown that X satisfies the non-strict Opial condition if and only if whenever (x_n) converges weakly to a non-zero limit x_∞, for $x_n^* \in J(x_n)$ we have

$$\liminf_n x_n^*(x_\infty) \geq 0.$$

Assume that $\psi_{(x_n)}(tx)$ is not an increasing function of t then there exists $s, t \in [0, \infty)$ such that $s > t$ and $\psi_{(x_n)}(sx) < \psi_{(x_n)}(tx)$. That is

$$\liminf_{x\to\infty} \|sx - x_n\| < \liminf_{x\to\infty} \|tx - x_n\|$$

Let $x_n^* \in J(tx - x_n)$ which means $x_n^*(tx - x_n) = \|tx - x_n\|^2$ and $\|x_n^*\| = \|tx - x_n\|$.

Then

$$\begin{aligned}\liminf_{x\to\infty} x_n^*(sx - x_n) &\leq \liminf_{x\to\infty} \|x_n^*\| \liminf_{x\to\infty} \|sx - x_n\| \\ &< \liminf_{x\to\infty} \|x_n^*\| \liminf_{x\to\infty} \|tx - x_n\| \\ &= \liminf_{x\to\infty} \|tx - x_n\|^2 \\ &= \liminf_{x\to\infty} x_n^*(tx - x_n)\end{aligned}$$

So

$$s \liminf_{x\to\infty} x_n^*(x) - \limsup_{x\to\infty} x_n^*(x_n) < t \liminf_{x\to\infty} x_n^*(x) - \limsup_{x\to\infty} x_n^*(x_n)$$

Using $\liminf_n x_n^*(tx) \geq 0$ this last strict inequality means $s < t$, a contradiction.

4. Conclusion

This paper presents a small result associated with research into the fixed point property in Banach spaces. The chief aim here is to use this result as a vehicle to illustrate Brailey's research efforts. Looking at the reference list it is clear that Brailey has been innovative in his own right and successfully collaborated with several leading researchers. He was very helpful and generous in guiding his research student, me, into research and publication.

Thank you Brailey. I hope there are many more years of research and collaboration.

References

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