Semigroup C*-algebras. Ideal structure, classification, and outlook

Xin Li

Queen Mary University of London (QMUL)

Semigroup C*-algebras. The boundary quotient

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Let the two generators of $\mathbb{N} * \mathbb{N}$ be a and b. Their isometries V_a and V_b satisfy $V_a V_a^* \perp V_b V_b^*$. To get $\mathcal{O}_2$, we must have $V_a V_a^* + V_b V_b^* = 1$.

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Let the two generators of $\mathbb{N} * \mathbb{N}$ be a and b. Their isometries V_a and V_b satisfy $V_a V_a^* \perp V_b V_b^*$. To get $\mathcal{O}_2$, we must have $V_a V_a^* + V_b V_b^* = 1$.

For $P = \mathbb{N} * \mathbb{N}$, Ω_P can be identified with the space of all finite and infinite words in a and b.

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Let the two generators of $\mathbb{N} * \mathbb{N}$ be a and b. Their isometries V_a and V_b satisfy $V_a V_a^* \perp V_b V_b^*$. To get $\mathcal{O}_2$, we must have $V_a V_a^* + V_b V_b^* = 1$.

For $P = \mathbb{N} * \mathbb{N}$, Ω_P can be identified with the space of all finite and infinite words in a and b. $V_a V_a^*$ is the characteristic function on the subspace of all words starting with a, and similarly for $V_b V_b^*$.

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Let the two generators of $\mathbb{N} * \mathbb{N}$ be a and b. Their isometries V_a and V_b satisfy $V_a V_a^* \perp V_b V_b^*$. To get $\mathcal{O}_2$, we must have $V_a V_a^* + V_b V_b^* = 1$.

For $P = \mathbb{N} * \mathbb{N}$, Ω_P can be identified with the space of all finite and infinite words in a and b. $V_a V_a^*$ is the characteristic function on the subspace of all words starting with a, and similarly for $V_b V_b^*$.
$V_a V_a^* + V_b V_b^* = 1$ would mean that every word starts with either a or b.

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Let the two generators of $\mathbb{N} * \mathbb{N}$ be a and b. Their isometries V_a and V_b satisfy $V_a V_a^* \perp V_b V_b^*$. To get $\mathcal{O}_2$, we must have $V_a V_a^* + V_b V_b^* = 1$.

For $P = \mathbb{N} * \mathbb{N}$, Ω_P can be identified with the space of all finite and infinite words in a and b. $V_a V_a^*$ is the characteristic function on the subspace of all words starting with a, and similarly for $V_b V_b^*$.
$V_a V_a^* + V_b V_b^* = 1$ would mean that every word starts with either a or b.
But: There is the empty word.

Semigroup C*-algebras. The boundary quotient

Motivating example: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$. How can we go from $\mathcal{T}_2$ to $\mathcal{O}_2$?

Let the two generators of $\mathbb{N} * \mathbb{N}$ be a and b. Their isometries V_a and V_b satisfy $V_a V_a^* \perp V_b V_b^*$. To get $\mathcal{O}_2$, we must have $V_a V_a^* + V_b V_b^* = 1$.

For $P = \mathbb{N} * \mathbb{N}$, Ω_P can be identified with the space of all finite and infinite words in a and b. $V_a V_a^*$ is the characteristic function on the subspace of all words starting with a, and similarly for $V_b V_b^*$.
$V_a V_a^* + V_b V_b^* = 1$ would mean that every word starts with either a or b.
But: There is the empty word.

Idea: Remove the empty word. To obtain a quotient, we have to remove an invariant subspace, i.e., all finite words. We end up with the subspace of all infinite words. This is $\partial\Omega_P$.

Semigroup C*-algebras. The boundary quotient

Semigroup C*-algebras. The boundary quotient

For general $P \subseteq G$, $\partial\Omega_P = \overline{\Omega_{P,\max}}$ where $\Omega_{P,\max}$ are the maximal filters.

Semigroup C*-algebras. The boundary quotient

For general $P \subseteq G$, $\partial\Omega_P = \overline{\Omega_{P,\max}}$ where $\Omega_{P,\max}$ are the maximal filters. These are all characters $\chi : \mathcal{J}_P \to \{0,1\}$ for which $\chi^{-1}(1)$ is maximal.

Semigroup C*-algebras. The boundary quotient

For general $P \subseteq G$, $\partial\Omega_P = \overline{\Omega_{P,\max}}$ where $\Omega_{P,\max}$ are the maximal filters. These are all characters $\chi : \mathcal{J}_P \to \{0,1\}$ for which $\chi^{-1}(1)$ is maximal.

Definition

The boundary quotient of $C^*_\lambda(P)$ is given by

$$\partial C^*_\lambda(P) := C^*_\lambda(G \ltimes \partial\Omega_P) \cong C(\partial\Omega_P) \rtimes_r G.$$

Semigroup C*-algebras. The boundary quotient

For general $P \subseteq G$, $\partial\Omega_P = \overline{\Omega_{P,\max}}$ where $\Omega_{P,\max}$ are the maximal filters. These are all characters $\chi : \mathcal{J}_P \to \{0,1\}$ for which $\chi^{-1}(1)$ is maximal.

Definition

The boundary quotient of $C^*_\lambda(P)$ is given by

$$\partial C^*_\lambda(P) := C^*_\lambda(G \ltimes \partial\Omega_P) \cong C(\partial\Omega_P) \rtimes_r G.$$

$\partial\Omega_P$ is the smallest G-invariant closed subspace of Ω_P, in particular minimal, and $G \curvearrowright \partial\Omega_P$ is purely infinite if $\partial\Omega_P$ is not trivial.

Semigroup C*-algebras. The boundary quotient

For general $P \subseteq G$, $\partial\Omega_P = \overline{\Omega_{P,\max}}$ where $\Omega_{P,\max}$ are the maximal filters. These are all characters $\chi : \mathcal{J}_P \to \{0,1\}$ for which $\chi^{-1}(1)$ is maximal.

Definition

The boundary quotient of $C^*_\lambda(P)$ is given by

$$\partial C^*_\lambda(P) := C^*_\lambda(G \ltimes \partial\Omega_P) \cong C(\partial\Omega_P) \rtimes_r G.$$

$\partial\Omega_P$ is the smallest G-invariant closed subspace of Ω_P, in particular minimal, and $G \curvearrowright \partial\Omega_P$ is purely infinite if $\partial\Omega_P$ is not trivial.

Hence if $G \curvearrowright \partial\Omega_P$ is topologically free, then $\partial C^*_\lambda(P)$ will be a purely infinite simple C*-algebra.

The boundary quotient. Examples

The boundary quotient. Examples

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$.

The boundary quotient. Examples

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$.

Γ is co-irreducible, if we cannot find a non-trivial decomposition $V = V_1 \sqcup V_2$ such that $V_1 \times V_2 \in E$. If Γ is co-irreducible and not a singleton, $\partial C_\lambda^*(A_\Gamma^+)$ is a unital UCT Kirchberg algebra.

The boundary quotient. Examples

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E \rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$.

Γ is co-irreducible, if we cannot find a non-trivial decomposition $V = V_1 \sqcup V_2$ such that $V_1 \times V_2 \in E$. If Γ is co-irreducible and not a singleton, $\partial C_\lambda^*(A_\Gamma^+)$ is a unital UCT Kirchberg algebra.

Let R be an integral domain, and $P = R \rtimes R^\times$. What is the boundary quotient of $C_\lambda^*(R \rtimes R^\times)$?

The boundary quotient. Examples

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E \rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$.

Γ is co-irreducible, if we cannot find a non-trivial decomposition $V = V_1 \sqcup V_2$ such that $V_1 \times V_2 \in E$. If Γ is co-irreducible and not a singleton, $\partial C_\lambda^*(A_\Gamma^+)$ is a unital UCT Kirchberg algebra.

Let R be an integral domain, and $P = R \rtimes R^\times$. What is the boundary quotient of $C_\lambda^*(R \rtimes R^\times)$?

On $\ell^2 R$, define $U^b \delta_x = \delta_{b+x}$ for $b \in R$, $S_a \delta_x = \delta_{ax}$ for $a \in R^\times$. Set $\mathfrak{A}_r[R] := C^*(\{U^b, S_a \colon b \in R, a \in R^\times\} \subseteq \mathcal{L}(\ell^2 R)$.

The boundary quotient. Examples

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \text{ for all } (v,w) \in E \rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$.

Γ is co-irreducible, if we cannot find a non-trivial decomposition $V = V_1 \sqcup V_2$ such that $V_1 \times V_2 \in E$. If Γ is co-irreducible and not a singleton, $\partial C_\lambda^*(A_\Gamma^+)$ is a unital UCT Kirchberg algebra.

Let R be an integral domain, and $P = R \rtimes R^\times$. What is the boundary quotient of $C_\lambda^*(R \rtimes R^\times)$?

On $\ell^2 R$, define $U^b \delta_x = \delta_{b+x}$ for $b \in R$, $S_a \delta_x = \delta_{ax}$ for $a \in R^\times$. Set $\mathfrak{A}_r[R] := C^*(\{U^b, S_a \colon b \in R, a \in R^\times\} \subseteq \mathcal{L}(\ell^2 R)$.

We have a canonical isomorphism $\partial C_\lambda^*(R \rtimes R^\times) \cong \mathfrak{A}_r[R]$ if R is not a field, and in that case, these are again unital UCT Kirchberg algebras.

Classification results

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E \rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$.

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \text{ for all } (v, w) \in E \rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$. Let $\Gamma_i = (V_i, E_i)$ be the co-irreducible subgraphs of Γ.

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v, w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$. Let $\Gamma_i = (V_i, E_i)$ be the co-irreducible subgraphs of Γ.

Let $t(\Gamma)$ be the number of those Γ_i with $|V_i| = 1$.

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$. Let $\Gamma_i = (V_i, E_i)$ be the co-irreducible subgraphs of Γ.

Let $t(\Gamma)$ be the number of those Γ_i with $|V_i| = 1$.
For fixed Γ_i, and $n \in \mathbb{N}$, let K_n be the set of all complete subgraphs of Γ_i with $n+1$ vertices.

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$. Let $\Gamma_i = (V_i, E_i)$ be the co-irreducible subgraphs of Γ.

Let $t(\Gamma)$ be the number of those Γ_i with $|V_i| = 1$.
For fixed Γ_i, and $n \in \mathbb{N}$, let K_n be the set of all complete subgraphs of Γ_i with $n+1$ vertices. Set $\chi(\Gamma_i) := 1 - \sum_{n=0}^{\infty}(-1)^n|K_n|$.

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v\colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$. Let $\Gamma_i = (V_i, E_i)$ be the co-irreducible subgraphs of Γ.

Let $t(\Gamma)$ be the number of those Γ_i with $|V_i| = 1$.
For fixed Γ_i, and $n \in \mathbb{N}$, let K_n be the set of all complete subgraphs of Γ_i with $n+1$ vertices. Set $\chi(\Gamma_i) := 1 - \sum_{n=0}^{\infty}(-1)^n|K_n|$.
For $k \in \mathbb{Z}$, let $N_k(\Gamma)$ be the number of Γ_i with $\chi(\Gamma_i) = k$.

Classification results

Let $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ for all } (v,w) \in E\rangle^+$ be the RAAM for the graph $\Gamma = (V, E)$. Let $\Gamma_i = (V_i, E_i)$ be the co-irreducible subgraphs of Γ.

Let $t(\Gamma)$ be the number of those Γ_i with $|V_i| = 1$.
For fixed Γ_i, and $n \in \mathbb{N}$, let K_n be the set of all complete subgraphs of Γ_i with $n+1$ vertices. Set $\chi(\Gamma_i) := 1 - \sum_{n=0}^{\infty}(-1)^n|K_n|$.
For $k \in \mathbb{Z}$, let $N_k(\Gamma)$ be the number of Γ_i with $\chi(\Gamma_i) = k$.

Theorem (Eilers-L-Ruiz)

Let Γ and Λ be finite graphs. The following are equivalent:

1. $C_\lambda^*(A_\Gamma^+) \cong C_\lambda^*(A_\Lambda^+)$
2.
   - $t(\Gamma) = t(\Lambda)$
   - $N_k(\Gamma) + N_{-k}(\Gamma) = N_k(\Lambda) + N_{-k}(\Lambda)$ for all $k \in \mathbb{Z}$
   - $N_0(\Gamma) > 0$ or $\sum_{k>0} N_k(\Gamma) \equiv \sum_{k>0} N_k(\Lambda) \mod 2$.

Classification results

Theorem (L)

Let K and L be number fields with rings of algebraic integers R and S. Assume that K and L have the same number of roots of unity. If $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ then $\zeta_K = \zeta_L$.

Classification results

Theorem (L)

Let K and L be number fields with rings of algebraic integers R and S. Assume that K and L have the same number of roots of unity. If $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ then $\zeta_K = \zeta_L$.

In particular, for Galois extensions K, L with the same number of roots of unity, $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ if and only if $K \cong L$.

Classification results

Theorem (L)

Let K and L be number fields with rings of algebraic integers R and S. If there exists an isomorphism $C_\lambda^*(R \rtimes R^\times) \cong C_\lambda^*(S \rtimes S^\times)$ sending $D_\lambda(R \rtimes R^\times)$ to $D_\lambda(S \rtimes S^\times)$, then $\zeta_K = \zeta_L$ and $Cl_K \cong Cl_L$.

Here $D_\lambda(P) = C_\lambda^*(P) \cap \ell^\infty(P)$.

Outlook: Nuclearity and embeddings into amenable groups

We observed: If a semigroup P embeds into an amenable group, then $C^*_\lambda(P)$ is nuclear.

Outlook: Nuclearity and embeddings into amenable groups

We observed: If a semigroup P embeds into an amenable group, then $C^*_\lambda(P)$ is nuclear. What about the converse?

Outlook: Nuclearity and embeddings into amenable groups

We observed: If a semigroup P embeds into an amenable group, then $C^*_\lambda(P)$ is nuclear. What about the converse?

Question

Let P be a semigroup which embeds into a group. If $C^*_\lambda(P)$ is nuclear, does P embed into an amenable group?

Outlook: Nuclearity and embeddings into amenable groups

We observed: If a semigroup P embeds into an amenable group, then $C^*_\lambda(P)$ is nuclear. What about the converse?

Question

Let P be a semigroup which embeds into a group. If $C^*_\lambda(P)$ is nuclear, does P embed into an amenable group?

More concrete questions:

Do right-angled Artin monoids embed into amenable groups?

Outlook: Nuclearity and embeddings into amenable groups

We observed: If a semigroup P embeds into an amenable group, then $C^*_\lambda(P)$ is nuclear. What about the converse?

Question

Let P be a semigroup which embeds into a group. If $C^*_\lambda(P)$ is nuclear, does P embed into an amenable group?

More concrete questions:

Do right-angled Artin monoids embed into amenable groups?

Do Baumslag-Solitar monoids embed into amenable groups?

Outlook: K-theory

Does our K-theory formula really require the Toeplitz condition?
Does it require the Baum-Connes conjecture?

Outlook: K-theory

Does our K-theory formula really require the Toeplitz condition?
Does it require the Baum-Connes conjecture?

Question

Given a left cancellative right LCM monoid P, do we always have $K_*(C^*_\lambda(P)) \cong K_*(C^*_\lambda(P^*))$?

Outlook: Left vs right

Let P be a cancellative semigroup.

Outlook: Left vs right

Let P be a cancellative semigroup. Then we can construct C*-algebras $C_\lambda^*(P)$ and $C_\rho^*(P)$ attached to left and right regular representations.

Outlook: Left vs right

Let P be a cancellative semigroup. Then we can construct C*-algebras $C_\lambda^*(P)$ and $C_\rho^*(P)$ attached to left and right regular representations.

These algebras can be very different.

Outlook: Left vs right

Let P be a cancellative semigroup. Then we can construct C*-algebras $C^*_\lambda(P)$ and $C^*_\rho(P)$ attached to left and right regular representations.

These algebras can be very different.

However, their K-theories coincide in all our computations.

Outlook: Left vs right

Let P be a cancellative semigroup. Then we can construct C*-algebras $C^*_\lambda(P)$ and $C^*_\rho(P)$ attached to left and right regular representations.

These algebras can be very different.

However, their K-theories coincide in all our computations. Also, there is no example where one is nuclear and the other one is not.

Outlook: Left vs right

Let P be a cancellative semigroup. Then we can construct C*-algebras $C^*_\lambda(P)$ and $C^*_\rho(P)$ attached to left and right regular representations.

These algebras can be very different.

However, their K-theories coincide in all our computations. Also, there is no example where one is nuclear and the other one is not.

Task

Find a cancellative semigroup P for which $C^*_\lambda(P)$ and $C^*_\rho(P)$ differ in K-theory, or with respect to nuclearity.

Thank you very much for your attention!