C^*-Algebras generated by semigroups of partial isometries

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We denote the least upper bound of x, y in the left order as $x \vee_l y$ and in the right order as $x \vee_r y$.

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(G, P) is quasi-lattice ordered in the left order if and only if every pair $a, b \in P$ has a greatest right lower bound $a \wedge_r b$.

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Corollary

The following are equivalent:

- *(G, P) is doubly quasi-lattice ordered.*
- *Any pair $x, y \in G$ with a common left upper bound in P has a least common left upper bound in P and every pair $a, b \in P$ has a greatest left lower bound $a \wedge_l b$.*
- *Any pair $x, y \in G$ with a common left upper bound in P has a least common right upper bound in P and every pair $a, b \in P$ has a greatest right lower bound $a \wedge_l b$.*

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- $(\mathbb{Z}^2, \mathbb{N}^2)$ is a doubly quasi-lattice ordered group.
  $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.

$$(a, b) \vee (c, d) = (\max\{a, c\}, \max\{b, d\}).$$

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- Let $\mathbb{F}_2$ be the free group with generators $\{a, b\}$, and let $\mathbb{F}_2^+$ be the free semigroup. Then $(\mathbb{F}_2, \mathbb{F}_2^+)$ is a doubly quasi-lattice ordered group.

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- $(\mathbb{Q} \rtimes \mathbb{Q}^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is doubly quasi-lattice ordered. For $(m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N}^\times$ we have

$$(m, a) \vee_l (n, b) < \infty \Leftrightarrow (m + a\mathbb{N}) \cap (n + b\mathbb{N}) \neq \emptyset.$$

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$$(m, a) \vee_l (n, b) < \infty \Leftrightarrow (m + a\mathbb{N}) \cap (n + b\mathbb{N}) \neq \emptyset.$$

However, $(m, a) \vee_r (n, b) < \infty$ for all $(m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N}^\times$.

- Let $c, d \geq 0$ and

$$\mathrm{BS}(c,d) := \langle a, b | ab^c = b^d a \rangle.$$

Let $\mathrm{BS}(c,d)^+$ be the subsemigroup generated by $\{a, b, e\}$. Then $(\mathrm{BS}(c,d), \mathrm{BS}(c,d)^+)$ is a doubly quasi-lattice ordered group. **(Spielberg)**

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- If $d \geq 0$ then $\mathrm{BS}(1, -d)$ is a quasi-lattice ordered group (in the left order) but not a doubly quasi-lattice ordered group.

Covariant Partial Isometric Representations

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Definition

Let (G, P) be a doubly quasi-lattice ordered group. A *partial isometric representation* of P is a map $W : P \to A$ such that W_p is a partial isometry for all $p \in P$, $W_e = 1$ and $W_x W_y = W_{xy}$ for all $x, y \in P$.

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A partial isometric representation is *left-covariant* if it satisfies

$$W_x W_x^* W_y W_y^* = \begin{cases} W_{x \vee_l y} W_{x \vee_l y}^* & \text{if } x \vee_l y < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

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A partial isometric representation is *right-covariant* if it satisfies

$$W_x^* W_x W_y^* W_y = \begin{cases} W_{x \vee_r y}^* W_{x \vee_r y} & \text{if } x \vee_r y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

If a partial isometric representation is both left- and right-covariant we say that it is *covariant*.

Covariant representations properties

We can rewrite the covariance identities as:

$$W_x^* W_y = W_x^* W_{x \vee_l y} W_{y^{-1}(x \vee_l y)}^*$$

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Lemma

Let $W : P \to A$ be a covariant partial isometric representation. Any product of the form $W_{n_1} W_{n_2}^* W_{n_3} W_{n_4}^* \ldots$ where $n_i \in P$ is either 0 or may be expressed as $W_p^* W_q W_r^*$ for some $p, q, r \in P$ satisfying $p \leq_l q$ and $r \leq_r q$.

Analogue of Truncated shifts

Definition

Let $A \subset P$. Define $J^A : P \to B(\ell^2(a))$ by

$$J^A_p \epsilon_a = \begin{cases} \epsilon_{pa} & \text{if } pa \in A \\ 0 & \text{otherwise} \end{cases}.$$

Lemma

1. *$J^A_p J^A_q = J^A_{pq}$ if and only if for all $a, b \in A$ we have $\{x \in P : a \leq_r x \leq_r b\} \subseteq A$.*
2. *J^A is left-covariant if and only if, for all $a, b \in A$ with a common right upper bound in A, $a \wedge_r b \in A$.*
3. *J^A is right-covariant if and only if, for all $a, b \in A$ with a common right lower bound in A, $a \vee_r b \in A$.*

Direct sums of Truncated shifts

Let (G, P) be a doubly quasi-lattice ordered group.
For $a \in P$ let $I_a := \{x \in P : x \leq_r a\}$.
Let $\{\epsilon_x\}$ be an orthonormal basis for $\ell^2(I_a)$. Then
$J^a : P \to B(\ell^2(I_a))$ defined

$$J_p^a \epsilon_x = \begin{cases} \epsilon_{px} & \text{if } px \leq_r a \\ 0 & \text{otherwise} \end{cases}$$

is a covariant partial isometric representation.

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is a covariant partial isometric representation.
Let $J : P \to B(\oplus_{a \in P} \ell^2(I_a))$ be defined as $J_p = \oplus J^a_p$. Let $C^*(J)$ be the C^*-algebra generated by $\{J_p : p \in P\}$.

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Lemma

The set $S := \{J_p^* J_q J_r^* : p, q, r \in P, p \leq_l q, r \leq_r q\}$ is linearly independent and $\operatorname{span} S$ is a dense unital $*$-subalgebra of $C^*(J)$.

Universal Algebra

Proposition

There is a C^*-algebra $C^*(G, P)$ generated by partial isometries $\{v_p : p \in P\}$ which has the following property: for every covariant partial isometric representation $W : P \to A$ there is a unital homomorphism $\pi_W : C^*(G, P) \to A$ such that $\pi_W(v_p) = W_p$.

Faithful Representations of $C^*(G, P)$

When is $\pi_J : C^*(G, P) \to C^*(J)$ faithful?

Proposition

There is a norm-decreasing linear idempotent $E : C^*(G, P) \to \overline{\operatorname{span}}\{v_p^* v_p v_r v_r^* : p, r \in P\}$ *such that*

$$E(\sum \lambda_{p,q,r} v_p^* v_q v_r^*) = \sum \lambda_{p,pr,r} v_p^* v_{pr} v_r^*.$$

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Theorem

The homomorphism $\pi_J : C^*(G, P) \to C^*(J)$ *is faithful if and only if* (G, P) *is amenable.*

Faithful representations

Definition

Let $W : P \to A$ be a covariant partial isometric representation.
Let $L^W_{(x_1,x_2)} = W_{x_1} W^*_{x_1} W^*_{x_2} W_{x_2}$.
A covariant partial isometric representations $W : P \to A$ *sees all projections* if, for every finite set $F \subset P_r \times P_l$ and $(x_1, x_2) \notin F$ such that (x_1, x_2) is a lower bound for F, we have

$$\prod_{y \in F} (L^W_{(x_1,x_2)} - L^W_y) \neq 0.$$

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Theorem

Let (G, P) be an amenable group and let $W : P \to A$ be a covariant partial isometric representation. Further, let π_W be the corresponding homomorphism of $C^*(G, P)$. If W sees all projections then π_W is faithful.

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Theorem

Let (G, P) and (K, Q) be doubly quasi-lattice ordered groups with a controlled map $\phi : (G, P) \to (K, Q)$. If K is amenable then (G, P) is amenable and $C^*(G, P)$ is nuclear.

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