The semigroup of a higher rank graph

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Classification

Theorem (Kirchberg-Phillips)

Let A and B be a separable, simple, nuclear and purely infinite C^*-algebra in UCT, then

$$A \cong B \Leftrightarrow \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

Definition

A *C^*-algebra* is a subalgebra of $B(H)$ which is norm-closed and closed under adjoints.
A separable, simple, nuclear C^*-algebra A is *purely infinite* iff $A \cong A \otimes \mathcal{O}_\infty$.
A C^*-algebra A is *nuclear* iff $A \otimes_{\min} B \cong A \otimes_{\max} B$ for any C^*-algebra B.
A separable C^*-algebra *satisfies UCT* iff it is KK-equivalent to an abelian C^*-algebra.
The *Elliott invariant* of a C^*-algebra is its K-theory paired with traces.

Question

When is a simple k-graph C^*-algebra purely infinite?

k-graphs

Definition (Kumjian-Pask)

Let Λ be a countable small category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. Then (Λ, d) is a k-graph if it satisfies the factorization property: For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k = \mathrm{span}_{\mathbb{N}}\{e_1, \ldots, e_k\}$ s.t.

$$d(\lambda) = m + n$$

there exist unique $\mu, \nu \in \Lambda$ satisfying: $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \mathrm{Obj}(\Lambda)$, the set of vertices. An element $\lambda \in \Lambda^{e_i}$ is called an edge. For $\lambda : u \to v$ we write

$$s(\lambda) = u, \;\; r(\lambda) = v.$$

For a vertex v, set $v\Lambda^n := \{\lambda \in \Lambda^n : r(\lambda) = v\}$.

Drawing 2-graphs

For every 2-graph Λ, its 2-coloured directed graph or *skeleton* E_Λ is:

- draw a dot for each vertex
- draw an arrow from $s(\lambda)$ to $r(\lambda)$ for each edge λ
- colour the arrows: if $\lambda \in \Lambda^{e_1} = d^{-1}(e_1)$ colour its arrow blue, if $\lambda \in \Lambda^{e_2}$ colour its arrow red,

Recall, by factorisation property, for each $\lambda \in \Lambda^{e_1+e_2}$ there exist unique edges $e_\lambda, h_\lambda \in \Lambda^{e_1}$ and $f_\lambda, g_\lambda \in \Lambda^{e_2}$ s.t. $\lambda = e_\lambda f_\lambda = g_\lambda h_\lambda$.

- record $\mathcal{C}_\Lambda = \{(e_\lambda f_\lambda, g_\lambda h_\lambda) : \lambda \in \Lambda^{e_1+e_2}\}$

Using the equivalence relation $\sim_{\mathcal{C}_\Lambda}$ on E_Λ^* generated by $\mathcal{C}_\Lambda$ we recover Λ: $\Lambda \cong E_\Lambda^*/\sim_{\mathcal{C}_\Lambda}$.

Conversely, for any 2-coloured directed graph E and collection $\mathcal{C}$ s.t. each blue-red and red-blue path of length 2 appears precisely once in $\mathcal{C}$, then $E^*/\sim_{\mathcal{C}}$ is a 2-graph.

The k-graph C^*-algebra

Definition (Kumjian-Pask)

Let Λ be a row-finite k-graph with no sources (i.e., the set $v\Lambda^n$ is finite and non-empty for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$). Then $C^*(\Lambda)$ is the universal C^*-algebra generated by a Cuntz-Krieger Λ-family: a collection of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ s.t.

- $\{s_v : v \in \Lambda^0\}$ are mutually orthogonal projections,
- $s_\mu s_\nu = s_{\mu\nu}$ whenever $r(\nu) = s(\mu)$,
- $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all paths λ, and
- $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Examples

$C^*(1 \text{ vertex}, 1 \text{ edge}) \cong C(\mathbb{T})$, $C^*(1 \text{ vertex}, 2 \text{ edges}) \cong \mathcal{O}_2$.

Classification

Recall that for A and B separable, simple, nuclear, UCT and purely infinite, $A \cong B \iff \mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. Every k-graph C^*-algebra is separable, nuclear and in UCT (Kumjian-Pask, Tu). Simplicity is also characterized in terms of properties of the k-graph (Robertson-Sims).

Question

When is a simple k-graph C^*-algebra purely infinite?

Theorem (Kumjian-Pask-Raeburn)

Every simple 1-graph C^*-algebra $C^*(\Lambda)$ is either purely infinite or AF depending on if there is a loop or not.

Theorem (Pask-Raeburn-Rrdam-Sims)

The dichotomy of Kumjian-Pask-Raeburn fails for $k = 2$.

Dichotomy

Conjecture (Astrid an Huef)

Let $C^*(\Lambda)$ be a simple C^*-algebra of a row-finite k-graph Λ with no sources. Then $C^*(\Lambda)$ is either purely infinite or stably finite.

Theorem (Pask-Sims-S)

Let $C^*(\Lambda)$ be a simple C^*-algebra of a row-finite k-graph Λ with no sources. Suppose its semigroup $S(\Lambda)$ (to be defined) is almost unperforated. Then $C^*(\Lambda)$ is either purely infinite or stably finite.

Definition

A pre-ordered (abelian) semigroup S is *almost unperforated* if for every $n \in \mathbb{N}$, $x, y \in S$,

$$(n+1)x \leq ny \Rightarrow x \leq y.$$

The type semigroup of a k-graph

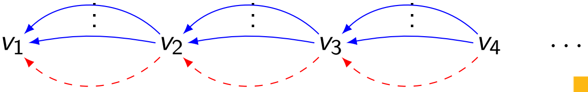
Definition

Let Λ be a row-finite k-graph with no sources. Denote the basis for $\mathbb{N}^k$ by $e_1, \ldots, e_k$ and let $\mathbb{N}\Lambda^0 := \operatorname{span}_{\mathbb{N}}\{\delta_v : v \in \Lambda^0\}$ be the abelian semigroup of finitely supported functions $f : \Lambda^0 \to \mathbb{N}$. We define

$$S(\Lambda) := \mathbb{N}\Lambda^0/{\approx_\Lambda}, \quad [f+g]_\Lambda := [f]_\Lambda + [g]_\Lambda$$

where $\approx_\Lambda$ is the smallest equivalence relation on $\mathbb{N}\Lambda^0$ making $S(\Lambda)$ into the semigroup s.t. $\delta_v \approx_\Lambda \sum_{\lambda \in v\Lambda^{e_i}} \delta_{s(\lambda)}$ for each $v \in \Lambda^0$ and $1 \leq i \leq k$.

Example



The type semigroup of a k-graph

Definition

Let S be an (abelian) semigroup with identity. We say that $x \in S$ is *infinite* if $x + y = x$ for some nonzero $y \in S$. We call S *stably finite* if it contains no infinite elements. If S is pre-ordered it is *purely infinite* if $2x \leq x$ for each nonzero $x \in S$.

Theorem (Pask-Sims-S, cf. Clark-an Huef-Sims)

Let $C^*(\Lambda)$ be a simple C^*-algebra of a row-finite k-graph Λ with no sources. Then

- *If $S(\Lambda)$ is stably finite or purely infinite then so is $C^*(\Lambda)$.*
- *If $k = 1$ then $S(\Lambda)$ is stably finite or purely infinite.*
- *$S(\Lambda)$ is stably finite $\Leftrightarrow$ $C^*(\Lambda)$ is stably finite.*
- *$S(\Lambda)$ is purely infinite $\Leftrightarrow$ $C^*(\Lambda)$ is purely infinite and $S(\Lambda)$ is almost unperforated.*

When is $S(\Lambda)$ almost unperforated?

Definition

Recall a pre-ordered (abelian) semigroup S is *almost unperforated* if for every $n \in \mathbb{N}$, $x, y \in S$,

$$(n+1)x \leq ny \Rightarrow x \leq y.$$

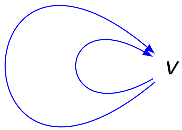
Theorem (Pask-Sims-S)

Let $C^*(\Lambda)$ be a simple C^*-algebra of a row-finite k-graph Λ with no sources. Then

- *If Λ has finitely many vertices, then $S(\Lambda)$ is almost unperforated.*
- *If Λ is strongly connected then $S(\Lambda)$ is almost unperforated.*
- *If Λ contains a cycle λ satisfying $d(\lambda) \geq (1, \ldots, 1)$, then $S(\Lambda)$ is almost unperforated.*

More Examples

Examples



More Examples

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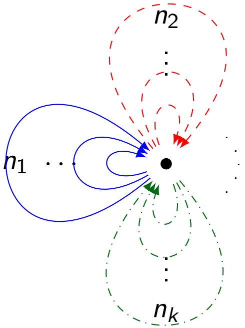

More Examples

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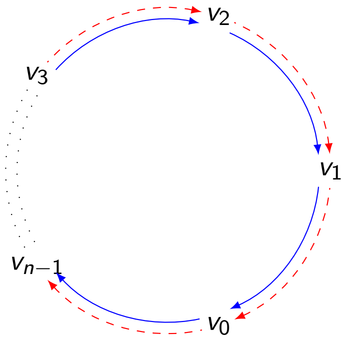

Thank you.

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