The dynamics of monotone vector inequalities

Björn S. Rüffer

The University of Newcastle & CARMA
bjorn.ruffer@newcastle.edu.au

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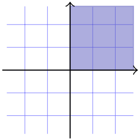
This talk is largely based on the paper:

> *B. S. Rüffer and R. Sailer. Input-to-state stability for discrete-time monotone systems. In Proc. 21st Int. Symp. Mathematical Theory of Networks and Systems (MTNS), pages 96–102, 2014.*

Order and monotonicity

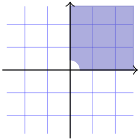
Partial ordering on $\mathbb{R}^n$

$$x \geq y \iff x_i \geq y_i \text{ for } i = 1, \ldots, n,$$



$$x > y \iff x \geq y \text{ and } x \neq y,$$

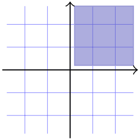
$$x \gg y \iff x_i > y_i \text{ for } i = 1, \ldots, n,$$



Monotone mapping

$g\colon \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$ monotone if

$$s \leq \tilde{s},\ w \leq \tilde{w} \quad \Rightarrow \quad g(s, w) \leq g(\tilde{s}, \tilde{w})$$



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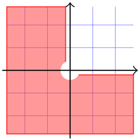

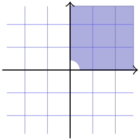

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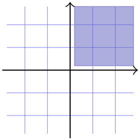


Basic notions

Discrete-time dynamical systems

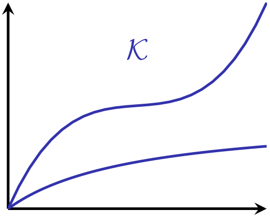
$$x^+ = g(x, u)$$

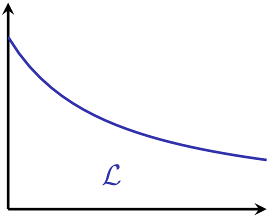
with $g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous, $g(0,0) = 0$.

Input-to-state stability

$$\|x[k]\| \leq \beta(\|x[0]\|, k) + \gamma(\|u\|_\infty)$$

where $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$.





Motivation I

Theorem (Perron)

Let A a positive $n \times n$ matrix. Then

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2. *Any nonnegative eigenvector of* A *is a multiple of* v.
3. *Any eigenvalue* $\lambda \neq \rho(A)$ *of* A *satisfies* $|\lambda| < \rho(A)$.

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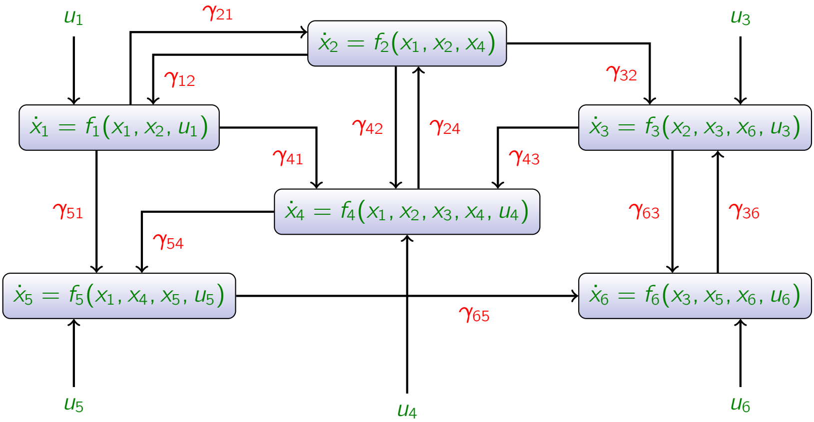
- $Ax \not\geq x$ for all $x > 0$;
- $Ax \geq x$ implies $x = 0$;
- there is a vector $v \gg 0$ so that $Av \ll v$;
- the inequality $x \leq Ax + b$ with $b \geq 0$ has the maximal solution $x = (I - A)^{-1} b \geq 0$

Motivation III: Large-scale systems



From subsystem stability to large-scale system stability

For each subsystem $\dot{x}_i = f_i(x_1, \ldots, x_n, u)$ we assume the existence of a continuous-time ISS Lyapunov function

$$V_i(x_i) \geq \sum_{j \neq i} \gamma_{ij}\big(V_j(x_j)\big) + \tilde{\gamma}(\|u\|) \quad \Rightarrow \quad \dot{V}_i < 0.$$

Modulo some technical details, if for each point $s \in \mathbb{R}^n_+$, $s \neq 0$,

$$\Gamma(s) \not\geq s \qquad \text{(small-gain condition)}$$

then the large-scale system $\dot{x} = f(x, u)$ is ISS.

[e.g. Jiang&Teel&Praly'96, Dashkovskiy&Rüffer&Wirth'10, Karafyllis&Jiang'09,...]

Aggregating Lyapunov functions

If there exist $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$ such that for all $r > 0$,

$$\Gamma(\sigma(r)) \ll \sigma(r),$$

then

$$\mathcal{V}(x) = \max_i \sigma_i^{-1}\big(V_i(x_i)\big)$$

is an ISS Lyapunov function for the composite large-scale system:

If $\mathcal{V}(x) = \sigma_i^{-1}\big(V_i(x_i)\big) > \max_{j \neq i} \sigma_j^{-1}\big(V_j(x_j)\big)$ for a unique i then

$$\begin{aligned} V_i = \sigma_i\big(\mathcal{V}\big) &> \Gamma_i\big(\sigma_1(\mathcal{V}), \ldots, \sigma_n(\mathcal{V})\big) \\ &= \Gamma_i\big(\sigma_1 \circ \sigma_i^{-1}(V_i), \ldots, \sigma_n \circ \sigma_i^{-1}(V_i)\big) \\ &\geq \Gamma_i(V_1, \ldots, V_n) \end{aligned}$$

so $\dot{V}_i < 0$ and hence $\dot{\mathcal{V}}(x) = (\sigma_i^{-1})'\big(V_i(x)\big)\,\dot{V}_i(x) < 0$.

Trajectory estimates

Individual ISS trajectory estimates

$$\|x_i(t)\| \leq \beta(\|x_i(0)\|, t) + \sum_{j \neq i} \gamma_{ij}(\|x_j\|_\infty) + \tilde{\gamma}(\|u\|_\infty)$$

leads to the vector-"matrix" inequality

$$s \leq \Gamma(s) + w.$$

Proving ISS of the composite large-scale system amounts to finding bounds of the form

$$\|s\| \leq \zeta(\|w\|).$$

[Dashkovskiy&Rüffer&Wirth'07]

Monotone systems

Let $g\colon \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$ be continuous and monotone, $g(0,0)=0$, then we call

$$x^+ = g(x,u)$$

a montone system.

For constant input u we write

$$g_u^k(x) = g(g(g(\dots,u),u),u).$$

For the remainder of the talk:

We consider a continuous monotone map

$$g\colon \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$$

with $g(0,0) = 0$ and the induced monotone dynamical system

$$x^+ = g(x,u).$$

The map g is called eventually increasable if for all $x \in \mathbb{R}^n_+$ there exists a $k \geq 1$ and $u \in \mathbb{R}^m_+$ such that

$$x \leq g^k_u(x). \tag{1}$$

A continuous monotone function $\zeta\colon \mathbb{R}^n_+ \to \mathbb{R}^m_+$ is called proper if there exists a function $\alpha \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n_+$,

$$\alpha(\|x\|)e \leq \zeta(x).$$

ISS-LF

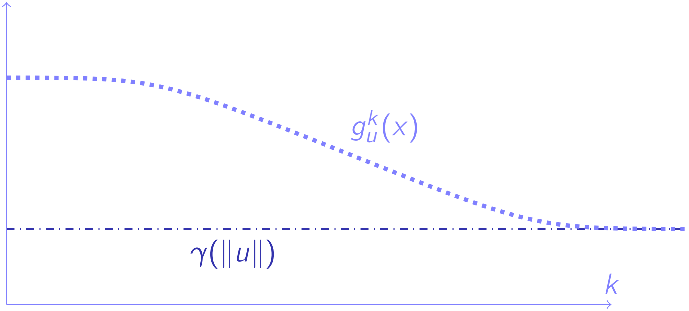
A continuous function $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ is an ISS Lyapunov function for $x^+ = g(x,u)$ if

- $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ and
- $V(x) \geq \gamma(\|u\|) \quad \Rightarrow \quad V(g(x,u)) - V(x) \leq -\alpha_3(V(x))$.

AG

The system has the asymptotic gain (AG) property if there exists a $\gamma \in \mathcal{K}$ such that for all $x \in \mathbb{R}^n_+$ and $u \in \mathbb{R}^m_+$,

$$\limsup_{k\to\infty} \|g_u^k(x)\| \leq \gamma(\|u\|).$$



Robust stability

We call the system robustly stable (RS) if there exists a proper and positive definite map $\zeta\colon \mathbb{R}^n_+ \to \mathbb{R}^m_+$ so that the origin is globally asymptotically stable with respect to

$$x^+ = f(x) := g(x, \zeta(x)).$$

UOC

The system satisfies the uniform order condition (UOC) if there exists a proper and positive definite map $\zeta\colon \mathbb{R}_+^m \to \mathbb{R}_+^n$ such that

$$g(x,u) \ngeq x \text{ for all } x \nleq \zeta(u).$$

Example:

$$\Gamma(s) + w \ngeq s \text{ for all } s \nleq w$$

for $w = 0$ this reduces to

$$\Gamma(s) \ngeq s \text{ for all } s > 0$$

NP

The system satisfies the Neumann property (NP) if there exists a proper and positive definite $\zeta\colon \mathbb{R}^m_+ \to \mathbb{R}^n_+$ such that for all $x \in \mathbb{R}^n_+$, $u \in \mathbb{R}^m_+$,

$$x \leq g(x,u) \quad \Rightarrow \quad x \leq \zeta(u).$$

Example:

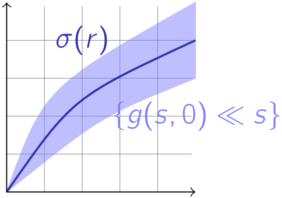
$x \leq Ax + b$ with A nonnegative and $\rho(A) < 1$, then

$$x \leq (I - A)^{-1} b$$
$$= (I + A + A^2 + A^3 + \ldots) b.$$

ΩP

The system satisfies the Ω path property (ΩP) if there exist proper and positive definite $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ and $\rho\colon \mathbb{R}_+ \to \mathbb{R}^m_+$ such that

$$\text{for all } r > 0, \quad g\big(\sigma(r), \rho(r)\big) \ll \sigma(r).$$



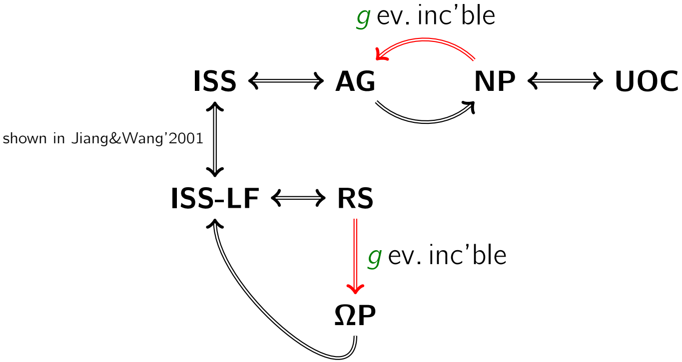
Examples:

$$Av \ll v$$

$$\Gamma\big(\sigma(r)\big) \ll \sigma(r)$$

Theorem

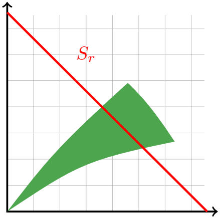
For a discrete-time monotone system all these system theoretic properties are essentially the same as ISS:



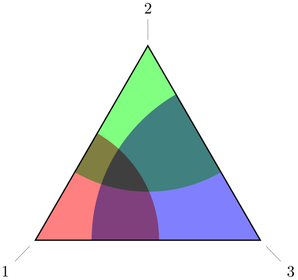
RS to ΩP

Sketch of the proof:

1. GAS of a monotone system $x^+ = f(x)$ implies that $f(x) \not\geq x$ for all $x > 0$.
2. This implies the existence of a path σ, s.t. $\sigma(0) = 0$, the components non-decreasing, at least one of them unbounded and $f(\sigma(r)) \ll \sigma(r), \quad \text{for } r > 0$ as per:
   - $\Omega_i := \{x \in \mathbb{R}^n_+ \colon f(x)_i < x_i\}$
   - $S_r := \{x \in \mathbb{R}^n_+ \colon \sum_i x_i = r\}$
   - $f(x) \not\geq x \Rightarrow \bigcup_{i=1}^n \Omega_i = \mathbb{R}^n_+ \setminus \{0\}$



3. KKM-Lemma: for all $r > 0$ the intersection $\bigcap_{i=1}^{n} \Omega_i \cap S_r$ is non-empty.



4. If f is proper, i.e., $f(x) \geq \alpha(\|x\|)e$, then all components of σ are unbounded.

5. RS of $x^+ = g(x, u)$ means RS of $x^+ = f(x) = g(x, \zeta(x))$. If σ is an Ω-path for f, then $\rho \coloneqq \zeta \circ \sigma$ satisfies $g(\sigma(r), \rho(r)) \ll \sigma(r)$.

Thank you!