Platonic lattices for trivalent Platonic polygonal complexes

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Special Session on Group Actions
AustMS 2012

Outline

1. Locally compact groups and lattices
2. Tree lattices
3. Platonic complexes and their lattices

Locally compact groups

G locally compact topological group with Haar measure μ

Examples

1. $G = (\mathbb{R}^n, +)$ with Lebesgue measure
2. $G = \mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$

Lattices

G locally compact, Haar measure μ

A subgroup $\Gamma < G$ is a lattice if

- Γ is discrete
- $\mu(\Gamma \backslash G) < \infty$ (finite covolume)

A lattice $\Gamma < G$ is

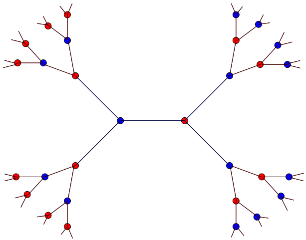
- uniform (or cocompact) if $\Gamma \backslash G$ is compact
- otherwise, nonuniform (or noncocompact)

Examples

1. $\mathbb{Z}^n$ is a uniform lattice in $\mathbb{R}^n$
2. $\mathrm{SL}(2, \mathbb{Z})$ is a nonuniform lattice in $\mathrm{SL}(2, \mathbb{R})$

Automorphism groups of trees

T locally finite tree e.g. T_3 the 3–regular tree



$G = \mathrm{Aut}(T)$, with compact-open topology, is a totally disconnected locally compact group.

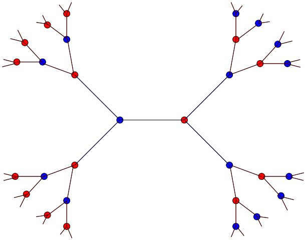
G nondiscrete $\iff \exists \{g_n\} \subset G \backslash \{1\}$ s.t. g_n fixes $\mathrm{Ball}(n)$.

Example

$G = \mathrm{Aut}(T_3)$ nondiscrete.

Motivation

- Study real Lie groups via action on symmetric space
  e.g. upper half-plane is symmetric space for $\mathrm{SL}(2, \mathbb{R})$
- Study "p–adic Lie groups" via action on building
  e.g. T_{q+1} is building for $\mathrm{SL}(2, \mathbb{F}_q((t)))$



Lattices in Aut(T)

T locally finite tree, $G = \mathrm{Aut}(T)$

$\Gamma < G$ is discrete $\iff$ Γ acts with finite stabilisers

Theorem (Serre)

Can normalise Haar measure μ on G so that $\forall$ discrete $\Gamma < G$

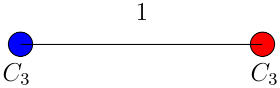
$$\mu(\Gamma \backslash G) = \sum_{v \in Vert(\Gamma \backslash T)} \frac{1}{|Stab_\Gamma(v)|} \leq \infty$$

and Γ cocompact $\iff$ $\Gamma \backslash T$ compact.

Examples of tree lattices

Cocompact lattice in $G = \mathrm{Aut}(T_3)$

$\Gamma = \pi_1(\text{graph of groups}) \cong C_3 * C_3$

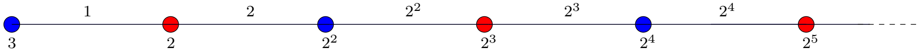


Examples of tree lattices

Non-cocompact lattice in $G = \mathrm{Aut}(T_3)$

$\Gamma = \pi_1(\text{graph of groups}) \cong C_3 * (\cdots)$

$\mu(\Gamma \backslash G) = \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{4}{3}$



Polygonal complexes

X locally finite, simply-connected polygonal complex

$G = \mathrm{Aut}(X)$ is locally compact group

Lattices $\Gamma \leq G$ characterised in same way as tree lattices:

$$\mu(\Gamma \backslash G) = \sum_{v \in \mathrm{Vert}(\Gamma \backslash X)} \frac{1}{|\mathrm{Stab}_\Gamma(v)|} \leq \infty$$

and Γ cocompact $\iff$ $\Gamma \backslash X$ compact.

Links and (k, L)–complexes

The link of a vertex v in X is the graph $L = \mathrm{Lk}(v, X)$ with

- Vert(L): edges of X incident at v
- Edge(L): faces of X incident at v
- Vertices adjacent in $L \iff$ corresp. edges of X share a face

Given $k \geq 3$ and a graph L, a (k, L)–complex is a polygonal complex X such that each face is a regular k–gon, and the link of each vertex is L.

Examples of simply-connected (k, L)–complexes

1. $k = 3$
   - L a hexagon: tessellation of $\mathbb{E}^2$ by equilateral triangles
   - L a generalised 3–gon: an $\tilde{A}_2$ building
2. $k = 4$
   - $L = K_{2,2}$: tessellation of $\mathbb{E}^2$ by squares
   - $L = K_{q,q}$: product of q-valent trees
3. $k \geq 5$
   - $L = K_{2,2}$: tessellation of $\mathbb{H}^2$ by right-angled k-gons
   - $L = K_{q,q}$: right-angled hyperbolic building
4. k even and L any simplicial graph: could be the Davis complex for Coxeter system

$$W = \langle S = \mathrm{Vert}(L) \mid s^2 = 1, (st)^{k/2} = 1 \iff s \text{ and } t \text{ adjacent}\rangle$$

Is $\mathrm{Aut}(X)$ nondiscrete?

- these $(k, K_{q,q})$–complexes have nondiscrete $\mathrm{Aut}(X)$ for $q \geq 3$.
- Davis complex has nondiscrete $\mathrm{Aut}(X)$ for L *flexible* (Haglund–Paulin, White)

Connection with δ–hyperbolic and CAT(0) square complexes

For $k \geq 4$, a (k, L)–complex X can be subdivided into a square complex, which is

- CAT(0) provided girth$(L) \geq 4$
- δ–hyperbolic provided girth$(L) \geq 5$

A uniform lattice Γ in $G = \mathrm{Aut}(X)$ is then a CAT(0) or word-hyperbolic group, respectively. In the latter case, by Agol's Theorem, Γ is virtually special hence linear.

Platonic complexes

A Platonic complex is a polygonal complex X such that $\mathrm{Aut}(X)$ acts transitively on flags (vertex, edge, face).

$\implies$ X is a (k, L)–complex with L an arc-transitive graph i.e. $\mathrm{Aut}(L)$ acts transitively on oriented edges of L.

Action on s–arcs

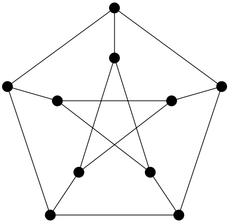
An s–arc in L is a tuple of vertices $(v_0, \ldots, v_s)$ s.t. v_i and v_{i+1} are adjacent and $v_{i-1} \neq v_{i+1}$. The graph L is s–arc transitive if $\mathrm{Aut}(L)$ acts transitively on the set of s–arcs, and s–arc regular if $\mathrm{Aut}(L)$ acts simply transitively on the set of s–arcs.

Theorem (Tutte 1947)

If L is a finite, connected, cubic and arc-transitive graph, then L is s–arc regular with $s \leq 5$.

Example

Petersen graph is 3–arc regular



Trivalent Platonic complexes

Theorem (Świątkowski 1999)

Let $k \geq 4$ and L be a finite, connected, arc-transitive cubic graph. If L is s–arc regular for $s \geq 3$ then $\exists$ a unique simply-connected (k, L)–complex X. Moreover X is Platonic and $\mathrm{Aut}(X)$ is nondiscrete.

Proof uses work of Djokovič–Miller, who classified finite, connected, arc-transitive cubic graphs L into 7 classes: L is s–arc regular for exactly one $s \in \{1, 2', 2'', 3, 4', 4'', 5\}$.

Platonic lattices

Let $X = X(k, L)$ be a trivalent Platonic complex as in Świątkowski's result.

Question

Does $G = \mathrm{Aut}(X)$ *admit a flag-transitive lattice?*
Equivalently, does G *admit a subgroup which acts flag-transitively with finite stabilisers?*

We call a flag-transitive lattice $\Gamma < G$ a Platonic lattice.

If Γ is a Platonic lattice, then its vertex stabilisers are finite and the induced action on the link of each vertex of X is that of an arc-transitive subgroup $H = H_\Gamma$ of $\mathrm{Aut}(L)$. Thus H is t–arc regular for some $t \leq s$.

Conder–Nedela refined the classification of Djokovič–Miller to include the values of $t < s$ such that $\mathrm{Aut}(L)$ admits a t–arc regular subgroup.

Results to date

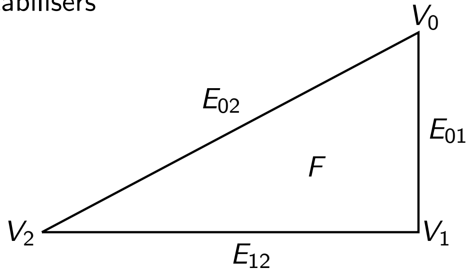
Theorem (Capdeboscq–Giudici–T 2012)

Let $k \geq 4$ and L be a finite, connected, arc-transitive cubic graph which is s–arc regular for $s \geq 3$. Let X be the unique simply-connected (k, L)–complex and let $G = \mathrm{Aut}(X)$.

1. *Consider H a t–arc regular subgroup of $\mathrm{Aut}(L)$.*
   1.1 *If $t \in \{1, 2'\}$, then for all k, the group G admits a Platonic lattice Γ with vertex stabilisers $\cong H$.*
   1.2 *If $t \in \{2'', 4', 4'', 5\}$ then G admits a Platonic lattice Γ with vertex stabilisers $\cong H$ if and only if k is even.*
   1.3 *If $t = 3$ then G admits a Platonic lattice Γ with vertex stabilisers $\cong H$ if and only if k is divisible by 2 or by 3.*
2. *If k is odd, there is no Platonic lattice $\Gamma < G$ such that the induced action on the link of each vertex is that of a $2''$–arc regular subgroup of $\mathrm{Aut}(L)$.*

Triangle of groups induced by action of Platonic lattice

Suppose Γ is a Platonic lattice. Then Γ acts on X with quotient a triangle in the barycentric subdivision of X, to which we may attach finite stabilisers



so that:

Link L' at V_0: for some $N \leq F$ with $N \lhd V_0$, and some t–arc transitive $H \leq \mathrm{Aut}(L)$,

$$V_0/N \cong H, \; E_{01}/N \cong \mathrm{Stab}_H(v), \; E_{02}/N \cong \mathrm{Stab}_H(e), \; F/N \cong \mathrm{Fix}_H(e)$$

Link $K_{2,3}$ at V_1: $|V_1 : E_{01}| = 2$, $|V_1 : E_{12}| = 3$ and $E_{01} \cap E_{12} = F$

Link $2k$–gon at V_2: $V_2/F \cong D_{2k}$ generated by $E_{12}/F \cong C_2$ and $E_{02}/F \cong C_2$

Triangles of groups

The theory of triangles of groups is due to Gersten and Stallings.

Graphs of groups	$\longleftrightarrow$	group actions on trees
Triangles of groups	$\longleftrightarrow$	group actions on triangle complexes
Complexes of groups	$\longleftrightarrow$	group actions on simplicial complexes, polyhedral complexes, scwols, ...

Proposition

There exists a Platonic lattice $\Gamma < \mathrm{Aut}(X)$ *if and only if there exists a triangle of finite groups as on the previous slide.*

Γ is the fundamental group and X is the universal cover of the triangle of groups.

Triangles of groups

A triangle of groups is developable if it is induced by a group action on a simply-connected triangle complex.
Not all triangles of groups are developable!

Theorem (Gersten–Stallings)

A nonpositively curved triangle of groups is developable.

Proposition

There exists a Platonic lattice $\Gamma < \mathrm{Aut}(X)$ *if and only if there is a triangle of finite groups as above.*

Proof.

Such a triangle of groups is developable since nonpositively curved, and the universal cover is the unique simply-connected (k, L)–complex X by Świątkowski's theorem. □

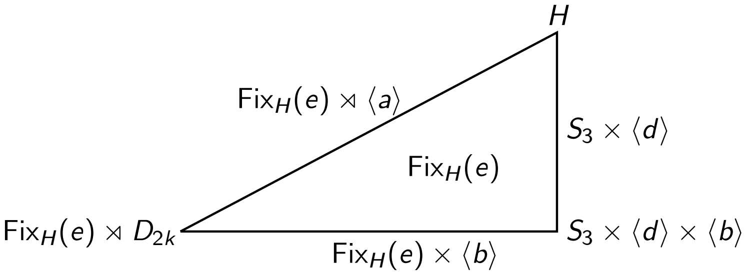
Example: 3–arc regular vertex stabilisers, k even

Suppose H is 3–arc regular e.g. $H = \mathrm{Aut}(L) \cong S_5$ for L the Petersen graph.

Then

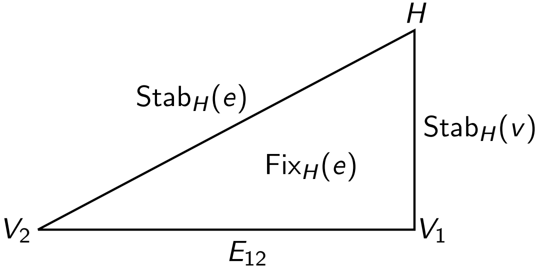
- $\mathrm{Fix}_H(e) = C_2 \times C_2 = \langle c \rangle \times \langle d \rangle$
- $\mathrm{Stab}_H(e) = D_8 = (\langle c \rangle \times \langle d \rangle) \rtimes \langle a \rangle$
- $\mathrm{Stab}_H(v) = S_3 \times C_2 = S_3 \times \langle d \rangle$

For k even, a triangle of groups for a Platonic lattice Γ is:



Example where Γ with vertex stabilisers $\cong H$ does not exist

Suppose $k \geq 5$ is odd and H is t–arc regular for $t \in \{2'', 4', 4'', 5\}$. Assume $\exists$ a Platonic lattice Γ with vertex stabilisers $\cong H$. Then Γ induces



with $\mathrm{Fix}_H(e)$ a 2–group.

Since k is odd, Sylow's theorems imply $E_{12} \cong \mathrm{Stab}_H(e)$ are Sylow 2–subgroups of V_2.

Now $|V_1 : E_{12}| = 3$ so V_1 is a group of order $3|\mathrm{Stab}_H(e)|$ with Sylow 2–subgroups isomorphic to $\mathrm{Stab}_H(e)$, and V_1 has an index 2 subgroup $\mathrm{Stab}_H(v)$. But in each case no such group V_1 exists.